

REPRESENTING KIRCHBERG ALGEBRAS AS INVERSE SEMIGROUP CROSSED PRODUCTS

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ABSTRACT. In this note we show that a combinatorial model of Kirchberg algebras in the UCT, namely the Katsura algebras $\mathcal{O}_{A,B}$, can be expressed both as groupoid C^* -algebras and as inverse semigroup crossed products. We use this picture to obtain results about simplicity, pure infiniteness and nuclearity of $\mathcal{O}_{A,B}$ using different methods than those used by Katsura.

1. INTRODUCTION

C^* -algebras have been, from its very origin in the 1940s, a class of operator algebras deeply studied and analyzed. The complexity of these objects forced the search of classifying invariants to analyze them, mostly of K-Theoretic nature [3].

In 1973 Elliott fixed the Classification Program: to classify separable nuclear C^* -algebras via K-Theoretic invariants (see [25] for an exhaustive account). This is still a very active field of research, with a great success in the case of purely infinite simple C^* -algebras. This class of algebras, originally introduced by Cuntz [7], have its most essential examples in the so-called Cuntz-Krieger algebras \mathcal{O}_A (where $A \in M_n(\mathbb{Z}^+)$) [8], a proper subclass of separable nuclear purely infinite simple C^* -algebras. Cuntz-Krieger algebras have a pure combinatorial algebraic nature, and its classification, obtained by Rørdam [24], is related to symbolic dynamics via Franks' classification of irreducible essential subshifts of finite type [14] (modulo the use of some sophisticated K-Theoretic tools). Simultaneously to Rørdam's work, Kirchberg [17] and Phillips [20] independently showed that separable nuclear purely infinite simple C^* -algebras satisfying the UCT (now called Kirchberg algebras) are classifiable using only K_0 and K_1 as invariants. This result extends Rørdam's one, but it is existential, and cannot be transferred to a combinatorial, algebraic context.

On the combinatorial framework Kumjian, Pask, Raeburn and Renault [18, 19] extended Cuntz-Krieger algebras to a biggest class, that of graph C^* -algebras (see [21] for an exhaustive account). The milestone of this class is that its properties are faithfully reflected in the combinatorial structure of the graphs used to construct it. In the purely infinite simple case, they conform a class extending that of purely infinite simple Cuntz-Krieger algebras, but it is still a proper subclass of Kirchberg algebras. In this context Sørensen [27] give a combinatorial classification result for simple graph C^* -algebras of graphs with finitely many vertices

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and countably many edges. This result and Rørdam's one combine to give a combinatorial classification result for unital simple graph C^* -algebras.

Finally, Katsura [15] constructed a class of algebras $\mathcal{O}_{A,B}$ (where $A \in M_n(\mathbb{Z}^+)$, $B \in M_n(\mathbb{Z})$) of combinatorial nature, generalizing Cuntz-Krieger algebras. Moreover, by using the Kirchberg-Phillips Theorem, he showed that any Kirchberg algebra can be represented (up to isomorphism) by an algebra of this family. Despite its purely combinatorial nature, Katsura analyzed these algebras via the notion of topological graph [16]. So, in the study of its structure and properties, the purely algebraic combinatorial framework is somehow forgotten. But these algebras clearly appear as a suitable combinatorial model for Kirchberg algebras, opening the possibility of an algebraic study of this relevant class of algebras.

Very recently, the first named author[9] developed a complete theory for studying C^* -algebras from a combinatorial point of view. In its early stages, his theory produced interesting approaches to Cuntz-Krieger algebras for infinite matrices [12] and more general objects [13]. Concretely, he analyzes how to represent large families of algebras via standardized constructions of combinatorial groupoids, and how to express these algebras as partial crossed products of commutative C^* -algebras by inverse semigroups and groups. In this way, he faithfully reflects the symbolic dynamic information associated to these algebras, and so opens the possibility of studying this kind of algebras by its intrinsic combinatorial structure, specially for its classification by combinatorial methods.

In this paper, we will study Katsura algebras from a combinatorial point of view, using the techniques developed by the first named author. At first sight, the more natural approach would be to mimic Exel and Laca's strategy [12], by searching a suitable group acting by partial automorphisms on a commutative C^* -algebra; in order to identify such a commutative algebra, the techniques developed in [13] would be essential. But the problem is that the natural group G associated to such a picture of $\mathcal{O}_{A,B}$ does not enjoy the right properties, so that the techniques of [13] do not apply to give us the desired description of $\mathcal{O}_{A,B}$ as a partial group crossed product. So, we need to start from a more basic point of view, by constructing a semigroupoid Λ whose properties allow us to use the techniques developed in [9] to present $\mathcal{O}_{A,B}$ as the C^* -algebra of Λ . By proving that Λ enjoys the right properties, we will then be able to present $\mathcal{O}_{A,B}$ as a groupoid C^* -algebra and as an inverse semigroup crossed product. The aim is to use this combinatorial point of view to associate a symbolic dynamical model to $\mathcal{O}_{A,B}$, and then to use it to state its basic properties in a more natural way. We are specially interested in stating the nuclearity of these algebras, and characterizing simplicity and pure infiniteness using this approach of the problem, more natural and intuitive. The future goal to be attended will be to understand the symbolic dynamics model associated to $\mathcal{O}_{A,B}$, in the search of a combinatorial, purely algebraic version of Kirchberg-Phillips Theorem.

The contents of this paper can be summarized as follows. In Section 2, we recall the definitions and results about Katsura algebras $\mathcal{O}_{A,B}$ we will use in the paper. Section 3 is devoted to constructing a suitable semigroupoid $\Lambda_{A,B}$ associated to $\mathcal{O}_{A,B}$, to analyze its basic properties and to define an inverse semigroup $\mathcal{S}(\Lambda_{A,B})$ associated to it. In Section 4 we state a tight representation of $\Lambda_{A,B}$ in $\mathcal{O}_{A,B}$, and we conclude that $\mathcal{O}_{A,B}$ is isomorphic to the full C^* -algebra of the tight groupoid $\mathcal{G}_{\Lambda_{A,B}}$ associated to $\mathcal{S}(\Lambda_{A,B})$; also, the amenability of the groupoid is proved. In Section 5 we compute an explicit form of $\mathcal{G}_{\Lambda_{A,B}}^{(0)}$, and thus we

obtain an explicit, operational presentation of the groupoid in terms of the action of the inverse semigroup $\mathcal{S}^{A,B}$ generated by the partial isometries of $\mathcal{O}_{A,B}$. By using this picture, we characterize when the groupoid is minimal and essentially free in Sections 6 and 7 respectively. In Section 8, under mild hypotheses, we obtain a characterization of simplicity for $\mathcal{O}_{A,B}$ improving Katsura's results, who only gives sufficient conditions for simplicity. In Section 9 we state sufficient conditions for the algebra $\mathcal{O}_{A,B}$ be purely infinite (simple). In the simple case, because of the characterization result obtained before, it extends the case stated by Katsura to more general situations.

2. SUMMARY ON KATSURA ALGEBRAS

In this section we will quickly recall the definition and basic properties of Katsura algebras that will be needed in the sequel. All the contents of this section are borrowed from [15].

Definition 2.1. Let $N \in \mathbb{N} \cup \{\infty\}$, let $A \in M_N(\mathbb{Z}^+)$ and $B \in M_N(\mathbb{Z})$ be row-finite matrices. Define a set Ω_A by

$$\Omega_A := \{(i, j) \in \{1, 2, \dots, N\} \times \{1, 2, \dots, N\} \mid A_{i,j} \geq 1\}.$$

For each $i \in \{1, 2, \dots, N\}$, define a set $\Omega_A(i) \subset \{1, 2, \dots, N\}$ by

$$\Omega_A(i) := \{j \in \{1, 2, \dots, N\} \mid (i, j) \in \Omega_A\}.$$

Notice that, by definition, $\Omega_A(i)$ is finite for all i . Finally, fix the following relation:

$$(0) \quad \Omega_A(i) \neq \emptyset \text{ for all } i, \text{ and } B_{i,j} = 0 \text{ for } (i, j) \notin \Omega_A.$$

Remark 2.2. Notice that the first statement on Condition (0) implies that A has no identically zero rows.

With these data we can define the algebras

Definition 2.3. With the data of Definition 2.1, define $\mathcal{O}_{A,B}$ to be the universal C^* -algebra generated by mutually orthogonal projections $\{q_i\}_{i=1}^N$, partial unitaries $\{u_i\}_{i=1}^N$ with $u_i u_i^* = u_i^* u_i = q_i$, and partial isometries $\{s_{i,j,n}\}_{(i,j) \in \Omega_A, n \in \mathbb{Z}}$ satisfying the relations:

- (i) $s_{i,j,n} u_j = s_{i,j,n+A_{i,j}}$ and $u_i s_{i,j,n} = s_{i,j,n+B_{i,j}}$ for all $(i, j) \in \Omega_A$ and $n \in \mathbb{Z}$.
- (ii) $s_{i,j,n}^* s_{i,j,n} = q_j$ for all $(i, j) \in \Omega_A$ and $n \in \mathbb{Z}$.
- (iii) $q_i = \sum_{j \in \Omega_A(i)} \sum_{n=1}^{A_{i,j}} s_{i,j,n} s_{i,j,n}^*$ for all i .

We will denote $p_{i,j,n} := s_{i,j,n} s_{i,j,n}^*$.

2.4. Now, the following facts holds:

- (1) The C^* -algebra $\mathcal{O}_{A,B}$ is separable, nuclear and in the UCT class [15, Proposition 2.9].
- (2) If the matrices A, B satisfy the following addition properties:
 - (a) A is irreducible.
 - (b) $A_{i,i} \geq 2$ and $B_{i,i} = 1$ for every $1 \leq i \leq N$.
 then the C^* -algebra $\mathcal{O}_{A,B}$ is simple and purely infinite, and hence a Kirchberg algebra [15, Proposition 2.10].
- (3) The K -groups of $\mathcal{O}_{A,B}$ are [15, Proposition 2.6]:
 - (a) $K_0(\mathcal{O}_{A,B}) \cong \text{coker}(I - A) \oplus \ker(I - B)$,
 - (b) $K_1(\mathcal{O}_{A,B}) \cong \text{coker}(I - B) \oplus \ker(I - A)$.

- (4) Every Kirchberg algebra can be represented, up to isomorphism, by an algebra $\mathcal{O}_{A,B}$ for matrices A, B satisfying the conditions (2)(a&b) [16, Proposition 4.5].

Besides that, it is useful to recall the following facts

Lemma 2.5. *Denote by I the multiindex set $\{(i_1, i_2, n_1)(i_2, i_3, n_2) \cdots (i_k, i_{k+1}, n_k)\}$, where $(i_j, i_{j+1}) \in \Omega_A$ and $n_j \in \mathbb{Z}$ for all j , and define $s(I) = i_1$, $t(I) = i_{k+1}$. Given a multiindex I as above, define $s_I := s_{i_1, i_2, n_1} s_{i_2, i_3, n_2} \cdots s_{i_k, i_{k+1}, n_k}$. Then:*

- (1) $\mathcal{O}_{A,B} = \overline{\text{span}}\{s_I s_J^* \mid I, J \text{ multiindices with } t(I) = t(J)\}$.
- (2) For any multiindex I there exists a unique multiindex \hat{I} such that $1 \leq \hat{n}_j \leq A_{i_j, i_{j+1}}$ for all j and a unique $t_I \in \mathbb{Z}$ such that $s_I = s_{\hat{I}} u_{k+1}^{t_I}$.
- (3) If $T^{A,B} = \{s_I u_{r(I)}^t \mid I \text{ multiindex in (2), } t \in \mathbb{Z}\}$ then the multiplicative sub-semigroup of $\mathcal{O}_{A,B}$, denote by $\mathcal{S}^{A,B}$ generated by $T^{A,B} \cup (T^{A,B})^*$ is an inverse semigroup with zero.

In the case that $B = (0)$ [15, Proposition 4.7], $\mathcal{O}_{A,(0)}$ is isomorphic to the Cuntz-Krieger algebra \mathcal{O}_A (the Exel-Laca algebra [12] if $N = \infty$) generated by mutually orthogonal projections $\{q_i\}_{i=1}^N$, and partial isometries $\{s_{i,j,n} \mid (i,j) \in \Omega_A, 1 \leq n \leq A_{i,j}\}$ satisfying the relations:

(i) $s_{i,j,n}^* s_{i,j,n} = q_j$ for all $(i,j) \in \Omega_A$ and $1 \leq n \leq A_{i,j}$.

(ii) $q_i = \sum_{j \in \Omega_A(i)} \sum_{n=1}^{A_{i,j}} p_{i,j,n}$ for all i .

where $p_{i,j,n} := s_{i,j,n} s_{i,j,n}^*$.

2.6. Notice that for any matrix B there is a natural injective $*$ -homomorphism $\mathcal{O}_A \hookrightarrow \mathcal{O}_{A,B}$ [15, Proposition 4.5]. When the multiindex sets $\{(i_1, i_2, n_1)(i_2, i_3, n_2) \cdots (i_k, i_{k+1}, n_k)\}$ restrict to the case where $(i_j, i_{j+1}) \in \Omega_A$ and $1 \leq n_j \leq A_{i_j, i_{j+1}}$ for all j , then $\mathcal{O}_A = \overline{\text{span}}\{s_I s_J^* \mid I, J \text{ multiindices with } t(I) = t(J)\}$. Moreover, if $T^A = \{s_I \mid I \text{ multiindex}\}$ then the semigroup \mathcal{S}^A generated by $T^A \cup T^{A*}$ is an inverse semigroup with zero, and the semilattices of projections $E(\mathcal{S}^A)$ and $E(\mathcal{S}^{A,B})$ coincide.

3. THE SEMIGROUPOID $\Lambda_{A,B}$

In this section we will associate to any pair of matrices A, B satisfying Definition 2.1 a semigroupoid $\Lambda_{A,B}$, and we will state the essential properties enjoyed by this semigroupoid. Also, we will briefly describe the inverse semigroup $\mathcal{S}(\Lambda_{A,B})$ associated to $\Lambda_{A,B}$ according to [9, Section 14]. The essential idea to be kept in mind is that our semigroupoid is a sort of path semigroupoid.

Definition 3.1. For matrices A, B satisfying Definition 2.1, fix a set

$$G := \{h_i \mid 1 \leq i \leq N\} \cup \{g_{i,j,n} \mid (i,j) \in \Omega_A, n \in \mathbb{Z}\}.$$

Then, we define $\Lambda_{A,B}$ to be the semigroupoid generated by G , satisfying:

- (1) The (partial) operation is induced by the admissible pairs

$$\{(h_i, g_{i,j,n}), (g_{i,j,n}, h_j), (g_{i,j,n}, g_{j,k,m}), (h_i, h_i)\}$$

for all $1 \leq i, j \leq N$, $(i, j), (j, k) \in \Omega_A$, $n, m \in \mathbb{Z}$.

- (2) $g_{i,j,n} h_j = g_{i,j,n+A_{i,j}}$ and $h_i g_{i,j,n} = g_{i,j,n+B_{i,j}}$ for all $(i, j) \in \Omega_A$ and $n \in \mathbb{Z}$.

Remark 3.2. Notice that, by the properties in Definition 3.1, the associative laws of the partial operation (see e.g. [9, Section 14]) hold, so that $\Lambda_{A,B}$ is a well-defined semigroupoid.

Now, we will show that there exists a standard form for the elements of $\Lambda_{A,B}$, and that this form is unique.

Lemma 3.3. $g_{i,j,n}g_{j,k,m} = g_{i,j,n-A_{i,j}}g_{j,k,m+B_{j,k}} = g_{i,j,n+A_{i,j}}g_{j,k,m-B_{j,k}}$

Proof. Simply notice that

$$g_{i,j,n-A_{i,j}}g_{j,k,m+B_{j,k}} = g_{i,j,n-A_{i,j}}h_jg_{j,k,m} = g_{i,j,n}g_{j,k,m}$$

and

$$g_{i,j,n+A_{i,j}}g_{j,k,m-B_{j,k}} = g_{i,j,n}h_jg_{j,k,m-B_{j,k}} = g_{i,j,n}g_{j,k,m}.$$

□

By Definition 3.1 and Lemma 3.3, it is easy to conclude that

Lemma 3.4. *The elements of $\Lambda_{A,B}$ have one of the following two forms:*

- (1) h_i^t for $1 \leq i \leq N$ and $t \in \mathbb{Z}$.
- (2) $g_{i_1,i_2,n_1}g_{i_2,i_3,n_2} \cdots g_{i_k,i_{k+1},n_k}$ for $(i_j, i_{j+1}) \in \Omega_A$, $n_j \in \mathbb{Z}$ for all j .

Moreover, we can prove the following result.

Lemma 3.5. *For any element $\alpha = g_{i_1,i_2,n_1}g_{i_2,i_3,n_2} \cdots g_{i_k,i_{k+1},n_k}$ with $(i_j, i_{j+1}) \in \Omega_A$, $n_j \in \mathbb{Z}$, there exists $1 \leq m_j \leq A_{i_j,i_{j+1}}$ for $1 \leq j \leq k-1$ and $m_k \in \mathbb{Z}$ such that*

$$\alpha = g_{i_1,i_2,m_1}g_{i_2,i_3,m_2} \cdots g_{i_{k-1},i_k,m_{k-1}}g_{i_k,i_{k+1},m_k}.$$

Proof. We will prove the result by induction on k . The case $k = 2$ being clear, let us suppose that the result is true for $l < k$, and let us prove it for $l = k$. Then, suppose that

$$\alpha = g_{i_1,i_2,n_1}g_{i_2,i_3,n_2} \cdots g_{i_{k-1},i_k,n_{k-1}}g_{i_k,i_{k+1},n_k}$$

with $(i_j, i_{j+1}) \in \Omega_A$, $n_j \in \mathbb{Z}$. By induction hypothesis, there exists $1 \leq m_j \leq A_{i_j,i_{j+1}}$ for $1 \leq j \leq k-2$ and $m'_{k-1} \in \mathbb{Z}$ such that

$$\alpha = g_{i_1,i_2,m_1}g_{i_2,i_3,m_2} \cdots g_{i_{k-2},i_{k-1},m_{k-2}}g_{i_{k-1},i_k,m'_{k-1}}g_{i_k,i_{k+1},n_k}.$$

Now, if $m'_{k-1} \geq 0$, there exist (unique) $1 \leq m_{k-1} \leq A_{i_{k-1},i_k}$ and $t \in \mathbb{Z}^+$ such that $m'_{k-1} = m_{k-1} + tA_{i_{k-1},i_k}$. Hence, by Lemma 3.3, $g_{i_{k-1},i_k,m'_{k-1}}g_{i_k,i_{k+1},n_k} = g_{i_{k-1},i_k,m_{k-1}}g_{i_k,i_{k+1},n_k+tB_{i_k,i_{k+1}}}$, so that the result holds by defining $m_k = n_k + tB_{i_k,i_{k+1}}$.

On the other hand, if $m'_{k-1} < 0$, there exist (unique) $1 \leq m_{k-1} \leq A_{i_{k-1},i_k}$ and $t \in \mathbb{Z}$ a negative integer such that $m'_{k-1} = m_{k-1} + tA_{i_{k-1},i_k}$. Then, by Lemma 3.3,

$$g_{i_{k-1},i_k,m'_{k-1}}g_{i_k,i_{k+1},n_k} = g_{i_{k-1},i_k,m'_{k-1}}g_{i_k,i_{k+1},n_k+|t|B_{i_k,i_{k+1}}-|t|B_{i_k,i_{k+1}}} =$$

$$g_{i_{k-1},i_k,m_{k-1}+tA_{i_{k-1},i_k}}h_{i_k}^{|t|}g_{i_k,i_{k+1},n_k-|t|B_{i_k,i_{k+1}}} = g_{i_{k-1},i_k,m_{k-1}}g_{i_k,i_{k+1},n_k-|t|B_{i_k,i_{k+1}}},$$

so that the result holds by defining $m_k = n_k - |t|B_{i_k,i_{k+1}} = n_k + tB_{i_k,i_{k+1}}$. □

We will call this special form of the elements of $\Lambda_{A,B}$ the *standard form*.

Lemma 3.6. *The standard form of an element of $\Lambda_{A,B}$ is unique.*

Proof. First, notice that the length of an element of $\Lambda_{A,B}$ (i.e. the number of $g_{i,j,n}$'s appearing in its expression), as well as the (i_j, i_{j+1}) 's involved, do not change by the defining relations. Thus, what we have to prove is that, whenever $\alpha \in \Lambda_{A,B}$ can be expressed both as

$$\alpha = g_{i_1, i_2, n_1} g_{i_2, i_3, n_2} \cdots g_{i_{k-1}, i_k, n_{k-1}} g_{i_k, i_{k+1}, n_k}$$

and as

$$\alpha = g_{i_1, i_2, m_1} g_{i_2, i_3, m_2} \cdots g_{i_{k-1}, i_k, m_{k-1}} g_{i_k, i_{k+1}, m_k}$$

with $1 \leq n_j, m_j \leq A_{i_j, i_{j+1}}$ for $1 \leq j \leq k-1$ and $n_k, m_k \in \mathbb{Z}$, then $n_j = m_j$ for all j .

By Lemma 3.3, there exist $r_1, \dots, r_{k-1}, s_1, \dots, s_{k-1} \in \mathbb{Z}$ such that:

- (1) $n_1 + r_1 A_{i_1, i_2} = m_1 + s_1 A_{i_1, i_2}$,
- (2) $n_j + r_j A_{i_j, i_{j+1}} - r_{j-1} B_{i_j, i_{j+1}} = m_j + s_j A_{i_j, i_{j+1}} - s_{j-1} B_{i_j, i_{j+1}}$ for all $2 \leq j \leq k-1$, and
- (3) $n_k - r_{k-1} B_{i_k, i_{k+1}} = m_k - s_{k-1} B_{i_k, i_{k+1}}$.

Now, notice that $n_1 + r_1 A_{i_1, i_2} = m_1 + s_1 A_{i_1, i_2}$ implies $n_1 - m_1 = (s_1 - r_1) A_{i_1, i_2}$. Since $1 \leq n_1, m_1 \leq A_{i_1, i_2}$, we get that $s_1 = r_1$ and thus $n_1 = m_1$.

Next, $n_2 + r_2 A_{i_2, i_3} - r_1 B_{i_2, i_3} = m_2 + s_2 A_{i_2, i_3} - s_1 B_{i_2, i_3}$, that by the previous computation implies $n_2 - m_2 = (s_2 - r_2) A_{i_2, i_3}$. Again, since $1 \leq n_2, m_2 \leq A_{i_2, i_3}$, we get that $s_2 = r_2$ and thus $n_2 = m_2$.

Recurrence on this argument shows that $s_j = r_j$ and $n_j = m_j$ for all j , as desired. \square

Recall from [9, Section 14] that given any semigroupoid Λ , we define $\tilde{\Lambda} = \Lambda \cup \{1\}$, and that an element $f \in \tilde{\Lambda}$ is said to be monic if for every $g, h \in \tilde{\Lambda}$ we have that $f g = f h$ implies $g = h$. Symmetrically, an element $f \in \tilde{\Lambda}$ is said to be epic if for every $g, h \in \tilde{\Lambda}$ we have that $g f = h f$ implies $g = h$.

Proposition 3.7. *Every element of $\Lambda_{A,B}$ is monic.*

Proof. We will write all the elements in standard form. Then, we need to distinguish various cases:

(1) $f = h_i^t, g = h_j^r, h = h_k^s$. Then, the definition of $\Lambda_{A,B}$ forces $i = j = k$, and since $f g = h_i^{t+r} = h_i^{t+s} = f h$, we conclude that $r = s$, i.e. $g = h$.

(2) $f = h_{i_1}^t$,

$$g = g_{i_1, i_2, n_1} g_{i_2, i_3, n_2} \cdots g_{i_{k-1}, i_k, n_{k-1}} g_{i_k, i_{k+1}, n_k}$$

and

$$h = g_{i_1, i_2, m_1} g_{i_2, i_3, m_2} \cdots g_{i_{k-1}, i_k, m_{k-1}} g_{i_k, i_{k+1}, m_k}.$$

Then,

$$f g = f h = g_{i_1, i_2, p_1} g_{i_2, i_3, p_2} \cdots g_{i_{k-1}, i_k, p_{k-1}} g_{i_k, i_{k+1}, p_k}$$

in standard form, which means that there exist $l_2, \dots, l_k, t_2, \dots, t_k \in \mathbb{Z}$ such that:

- (i) $p_1 = n_1 + l B_{i_1, i_2} - l_2 A_{i_1, i_2} = m_1 + l B_{i_1, i_2} - t_2 A_{i_1, i_2}$,
- (ii) $p_j = n_j + l_j B_{i_j, i_{j+1}} - l_{j+1} A_{i_j, i_{j+1}} = m_j + t_j B_{i_j, i_{j+1}} - t_{j+1} A_{i_j, i_{j+1}}$ for all $2 \leq j \leq k-1$,
and
- (iii) $p_k = n_k + l_k B_{i_k, i_{k+1}} = m_k + t_k B_{i_k, i_{k+1}}$.

The same argument as in the proof of Lemma 3.6 shows that $l_j = t_j$ and thus $n_j = m_j$ for all j , so that $g = h$.

(3) $f = g_{i_1, i_2, n_1} g_{i_2, i_3, n_2} \cdots g_{i_{k-1}, i_k, n_{k-1}} g_{i_k, i_{k+1}, n_k}$ in standard form, while $g = h_i^r, h = h_j^s$. Thus, we have $i = j = i_{k+1}$ and $n_k + r = n_k + s$, so that $r = s$ and thus $g = h$.

(4) $f = g_{j_1, j_2, n_1} g_{j_2, j_3, n_2} \cdots g_{j_{l-1}, j_l, n_{l-1}} g_{j_l, j_{l+1}, n_l}$, while

$$g = g_{i_1, i_2, m_1} g_{i_2, i_3, m_2} \cdots g_{i_{k-1}, i_k, m_{k-1}} g_{i_k, i_{k+1}, m_k}$$

and

$$h = g_{i_1, i_2, p_1} g_{i_2, i_3, p_2} \cdots g_{i_{k-1}, i_k, p_{k-1}} g_{i_k, i_{k+1}, p_k}$$

in standard form. By Lemma 3.6 there is no loss of generality in assuming that $l = 1$, and that $f = g_{i_0, i_1, n_0}$ with $n_0 \in \mathbb{Z}$. Now,

$$fg = fh = g_{i_0, i_1, q_0} g_{i_1, i_2, q_1} g_{i_2, i_3, q_2} \cdots g_{i_{k-1}, i_k, q_{k-1}} g_{i_k, i_{k+1}, q_k}$$

in normal form, so that there exist $l_1, \dots, l_k, t_1, \dots, t_k \in \mathbb{Z}$ such that:

- (i) $q_0 = n_0 - l_1 A_{i_0, i_1} = n_0 - t_1 A_{i_0, i_1}$,
- (ii) $q_j = m_j + l_j B_{i_j, i_{j+1}} - l_{j+1} A_{i_j, i_{j+1}} = p_j + t_j B_{i_j, i_{j+1}} - t_{j+1} A_{i_j, i_{j+1}}$ for all $1 \leq j \leq k-1$, and
- (iii) $q_k = m_k + l_k B_{i_k, i_{k+1}} = p_k + t_k B_{i_k, i_{k+1}}$.

Again the same argument as in the proof of Lemma 3.6 shows that $l_j = t_j$ and thus $m_j = p_j$ for all j , so that $g = h$. \square

By an analogue argument, we can show the characterization of epic elements in $\Lambda_{A,B}$. For this we need a definition.

Definition 3.8. We say that the matrix B satisfies Condition (E) when $B_{i,j} = 0$ if and only if $(i, j) \notin \Omega_A$.

Lemma 3.9. *Every element of $\Lambda_{A,B}$ is epic if and only if B satisfies Condition (E).*

3.10. Given two elements f, g in a semigroupoid Λ , recall that:

- (1) f divides g (or g is a multiple of f), denoted $f|g$, if either $f = g$ or there exists $h \in \Lambda$ such that $fh = g$.
- (2) f and g intersect if they admit a common multiple, i.e. there exists $m \in \Lambda$ such that both $f|m$ and $g|m$. We denote this relation by $f \bowtie g$.
- (3) f and g are disjoint if they do not intersect, and we denote it by $f \perp g$.
- (4) If $f, g \in \Lambda$ and $f \bowtie g$, we say that $m \in \Lambda$ is a least common multiple if it is a common multiple, and whenever $h \in \Lambda$ is a common multiple of f and g , then $m|h$. If it is unique, we will denote it by $m = \text{lcm}(f, g)$.

Then, we have the following result.

Proposition 3.11. *There exists a unique least common multiple for any pair of intersecting elements of $\Lambda_{A,B}$.*

Proof. We will assume that $f, g \in \Lambda$ and $f \bowtie g$, and we will prove the existence of a least common multiple for that pair. We need to distinguish various cases:

- (1) $f = h_i^t, g = h_j^r$. Then, if m is a common multiple, since either there exist $f', g' \in \Lambda_{A,B}$ such that $ff' = gg'$ or $m = f$ or $m = g$, the definition of $\Lambda_{A,B}$ forces $i = j$. Conversely, if

$i = j$, and we assume $t < r$ (when $t = r$, then $f = g$ and there is nothing to prove), then $fh_i^{r-t} = g$.

(2) $f = h_j^t$ and $g = g_{i_1, i_2, n_1} g_{i_2, i_3, n_2} \cdots g_{i_{k-1}, i_k, n_{k-1}} g_{i_k, i_{k+1}, n_k}$. As in case (1), since $f \mathbin{\frown} g$, we have that $j = i_1$. Conversely, if $j = i_1$, then

$$fg_{i_1, i_2, n_1 - tB_{i_1, i_2}} g_{i_2, i_3, n_2} \cdots g_{i_{k-1}, i_k, n_{k-1}} g_{i_k, i_{k+1}, n_k} = g.$$

Notice that the argument in (2) shows that:

- (i) In case (1) $m = h_i^r$ must be a least common multiple for f and g .
- (ii) In case (2) $m = g$ must be a least common multiple for f and g .

(3) $f = g_{i_1, i_2, n_1} g_{i_2, i_3, n_2} \cdots g_{i_{k-1}, i_k, n_{k-1}} g_{i_k, i_{k+1}, n_k}$ and $g = g_{j_1, j_2, m_1} g_{j_2, j_3, m_2} \cdots g_{j_{l-1}, j_l, m_{l-1}} g_{j_l, j_{l+1}, m_l}$, written in standard form. Since $f \mathbin{\frown} g$, there exist $f', g' \in \Lambda_{A, B}$ such that $ff' = gg'$. Hence, if $k \leq l$ then we have that $j_t = i_t$ for all $1 \leq t \leq k$ and $n_t = m_t$ for all $1 \leq t \leq k-1$. Moreover, define $\widehat{f} = g_{i_1, i_2, n_1} g_{i_2, i_3, n_2} \cdots g_{i_{k-1}, i_k, n_{k-1}}$, so that $f = \widehat{f} g_{i_k, i_{k+1}, n_k}$ and $g = \widehat{f} g_{i_k, i_{k+1}, m_k} g_{i_{k+1}, i_{k+2}, m_{k+1}} g_{i_{k+2}, i_{k+3}, m_{k+2}} \cdots g_{i_l, i_{l+1}, m_l}$. Since

$$\widehat{f} g_{i_k, i_{k+1}, n_k} f' = \widehat{f} g_{i_k, i_{k+1}, m_k} g_{i_{k+1}, i_{k+2}, m_{k+1}} g_{i_{k+2}, i_{k+3}, m_{k+2}} \cdots g_{i_l, i_{l+1}, m_l} g',$$

Lemma 3.3 implies that $n_k - m_k = tA_{i_k, i_{k+1}}$ for some $t \in \mathbb{Z}$. Conversely, if these restrictions hold, then:

- (i) If $k < l$, then

$$fg_{i_{k+1}, i_{k+2}, m_{k+1} - tB_{i_{k+1}, i_{k+2}}} g_{i_{k+2}, i_{k+3}, m_{k+2}} \cdots g_{i_l, i_{l+1}, m_l} = g,$$

so that $f \mathbin{\frown} g$ and g is a least common multiple for that pair.

- (ii) If $k = l$ and $f \neq g$, since $n_k = m_k + tA_{i_k, i_{k+1}}$ for some $t \in \mathbb{Z}$, we can assume that $t > 0$ (the other case is symmetric), and thus

$$gh_{i_{k+1}}^t = \widehat{f} g_{i_k, i_{k+1}, m_k} h_{i_{k+1}}^t = \widehat{f} g_{i_k, i_{k+1}, n_k} = f,$$

so that f is a least common multiple for the pair.

Uniqueness is a consequence of Proposition 3.7 (see [9, 14.8]), so we are done. \square

Remark 3.12. Notice that $\Lambda_{A, B}$ satisfies the Standing Hypothesis [9, 14.7] because of Proposition 3.7 and Proposition 3.11

Given an element f in a semigroupoid Λ , we define $\Lambda^f = \{g \in \Lambda \mid (f, g) \in \Lambda^{(2)}\}$. We say that an element is a spring if $\Lambda^f = \emptyset$. We say that Λ is categorical if given $f, g \in \Lambda$ then either $\Lambda^f = \Lambda^g$ or $\Lambda^f \cap \Lambda^g = \emptyset$. Then, we have the following result.

Lemma 3.13. *The semigroupoid $\Lambda_{A, B}$ has no springs and is categorical.*

Proof. Given any $f \in \Lambda_{A, B}$, f is either h_j^t or $\widehat{f} g_{i, j, n}$. Because of Definition 3.1, in both cases $\Lambda_{A, B}^f = \Lambda_{A, B}^{h_j}$. Then, $h_j \in \Lambda_{A, B}^f$, so that $\Lambda_{A, B}$ has no springs. The property of being categorical is then obvious. \square

Given a semigroupoid Λ , and $\Gamma \subseteq \Lambda$ a subset, we say that a subset $H \subset \Gamma$ is a cover for Γ if for every $f \in \Gamma$ there exists $h \in H$ such that $f \mathbin{\frown} h$. If moreover the elements of H are mutually disjoint then H is called a partition [9, Definition 15.3].

Proposition 3.14. *For every $f \in \Lambda_{A,B}$ and for every $h \in \Lambda_{A,B}^f$ there exists a finite partition $H \subseteq \Lambda_{A,B}^f$ such that $h \in H$.*

Proof. As noticed in the proof of Lemma 3.13, we can assume that $f = h_j$ for some $1 \leq j \leq N$. We will distinguish two cases:

(1) $h = h_j^t$ with $t \in \mathbb{N}$. Then, by the arguments in the proof of Proposition 3.11, every $g \in \Lambda_{A,B}^{h_j}$ intersects with h_i , so that it is enough to fix $H := \{h_j\}$.

(2) $h = g_{i_1,i_2,m_1}g_{i_2,i_3,m_2} \cdots g_{i_{k-1},i_k,m_{k-1}}g_{i_k,i_{k+1},m_k}$ (written in standard form) with $i_1 = j$. Then, $1 \leq m_j \leq A_{i_j,i_{j+1}}$ for $1 \leq j \leq k-1$ and $m_k \in \mathbb{Z}$, so that there exists exactly one $t \in \mathbb{Z}$ such that $m_k \in [tA_{i_k,i_{k+1}} + 1, (t+1)A_{i_k,i_{k+1}}]$ (i.e., $m_k = n_k + tA_{i_k,i_{k+1}}$ for a unique $1 \leq n_k \leq A_{i_k,i_{k+1}}$ and a unique $t \in \mathbb{Z}$). Now, we will construct the finite partition H using a “rooted tree” with “root” i_1 and paths constructed using as “edges” the pairs (i_j, i_{j+1}) appearing in the expression of h ; the idea is to use a minimal expansion of the root containing the path associated to h , and then complete this expansion in each vertex adding as many generators as $A_{i,j}$. Notice that, by hypothesis, fixed any i_j ($1 \leq j \leq k$), the set $\Omega_A(i_j)$ is finite, and thus so is the set of elements $p \in \mathbb{N}$ such that $1 \leq p \leq A_{i_j,l}$ for any $l \in \Omega_A(i_j)$. Now, we define H by recurrence, as follows:

- (i) Define $H_1 := \{g_{i_1,l,m} \mid l \in \Omega_A(i_1), 1 \leq l \leq A_{i_1,l}\} \setminus \{g_{i_1,i_2,m_1}\}$. Notice that, by the argument in the proof of Proposition 3.11, the elements of H_1 are mutually disjoint. Moreover, every element of $\Lambda_{A,B}^{h_{i_1}}$, except these elements starting in g_{i_1,i_2,m_1} , intersects with exactly one element of H_1 . Also, H_1 is finite.
- (ii) Define $H_2 := H_1 \cup (\{g_{i_1,i_2,m_1}g_{i_2,l,m} \mid l \in \Omega_A(i_2), 1 \leq m \leq A_{i_2,l}\} \setminus \{g_{i_1,i_2,m_1}g_{i_2,i_3,m_2}\})$. As in case (i), the elements of H_2 are mutually disjoint, and every element of $\Lambda_{A,B}^{h_{i_1}}$, except these elements starting in $g_{i_1,i_2,m_1}g_{i_2,i_3,m_2}$, intersects with exactly one element of H_2 . Also, H_2 is finite.
- (iii) Suppose defined H_j , $2 \leq j < k-1$, and define $H_{j+1} := H_j \cup (\{g_{i_1,i_2,m_1}g_{i_2,i_3,m_2} \cdots g_{i_j,l,m} \mid l \in \Omega_A(i_j), 1 \leq m \leq A_{i_j,l}\} \setminus \{g_{i_1,i_2,m_1}g_{i_2,i_3,m_2} \cdots g_{i_j,i_{j+1},m_j}\})$. As in the case (ii), the elements of H_{j+1} are mutually disjoint, and every element of $\Lambda_{A,B}^{h_{i_1}}$, except these elements starting in $g_{i_1,i_2,m_1}g_{i_2,i_3,m_2} \cdots g_{i_j,i_{j+1},m_j}$, intersects with exactly one element of H_{j+1} . Also, H_{j+1} is finite.
- (iv) Define $H := H_{k-1} \cup (\{g_{i_1,i_2,m_1}g_{i_2,i_3,m_2} \cdots g_{i_k,l,n+tA_{i_k,l}} \mid l \in \Omega_A(i_j), 1 \leq n \leq A_{i_k,l}\})$. As in case (iii), the elements of H are mutually disjoint, and every element of $\Lambda_{A,B}^{h_{i_1}}$ intersects with exactly one element of H . Also, H is finite, and $h \in H$.

So we are done. \square

3.15. Now, we will describe the inverse semigroup $\mathcal{S}(\Lambda_{A,B})$ associated to $\Lambda_{A,B}$. In order to define the elements of $\mathcal{S}(\Lambda_{A,B})$, by Lemma 3.13, we have

$$\mathcal{Q} = \{\Lambda_{A,B}^f \mid f \in \Lambda_{A,B}\} \cup \{\emptyset\} = \{\Lambda_{A,B}^{h_i} \mid 1 \leq i \leq N\} \cup \{\emptyset\}.$$

hence,

$$\begin{aligned} \mathcal{S}(\Lambda_{A,B}) = & \{(f, \Lambda_{A,B}^f, g) \mid f, g \in \Lambda_{A,B}\} \cup \{(f, \Lambda_{A,B}^f, 1) \mid f \in \Lambda_{A,B}\} \cup \{(1, \Lambda_{A,B}^g, g) \mid g \in \Lambda_{A,B}\} \\ & \cup \{(1, \Lambda_{A,B}^{h_i}, 1) \mid 1 \leq i \leq N\} \end{aligned}$$

with operation defined in [9, Definition 14.15]. Notice that this semigroup is naturally endowed with an involution via the formula $(f, A, g)^* := (g, A, f)$, and that it is an inverse

semigroup with zero [9, Theorem 14.16]. In fact, it is easy to see that $\mathcal{S}(\Lambda_{A,B})$ is generated (as semigroup) by the elements

$$\begin{aligned} & \{(1, \Lambda_{A,B}^{h_i}, h_i), (h_i, \Lambda_{A,B}^{h_i}, 1) \mid 1 \leq i \leq N\} \\ & \cup \{(g_{i,j,m}, \Lambda_{A,B}^{h_j}, 1), (1, \Lambda_{A,B}^{h_j}, g_{i,j,m}) \mid (i,j) \in \Omega_A, m \in \mathbb{Z}\}. \end{aligned}$$

Also, the semilattice $E(\mathcal{S}(\Lambda_{A,B}))$ of projections of $\mathcal{S}(\Lambda_{A,B})$ is easily described: if we define $p_f := (f, \Lambda_{A,B}^f, f)$ and $q_{h_i} := (1, \Lambda_{A,B}^{h_i}, 1)$, then

$$E(\mathcal{S}(\Lambda_{A,B})) = \{p_f \mid f \in \Lambda_{A,B}\} \cup \{q_i \mid 1 \leq i \leq N\}.$$

Moreover, the following relations hold (see [9, Proposition 19.4]):

- (i) $p_f p_g = p_{\text{lcm}(f,g)}$ if $f \cap g$.
- (ii) $p_f \leq p_g$ if and only if $g \mid f$.
- (iii) $p_f \perp p_g$ if $f \perp g$.
- (iv) $p_f \leq q_{h_i}$ if $f \in \Lambda_{A,B}^{h_i}$.
- (v) $p_f \perp q_{h_i}$ if $f \notin \Lambda_{A,B}^{h_i}$.
- (vi) $q_{h_i} \perp q_{h_j}$ if $i \neq j$.

Remark 3.16. By [9, Proposition 14.20], if every element in $\Lambda_{A,B}$ is epic then $\mathcal{S}(\Lambda_{A,B})$ is a E^* -unitary inverse semigroup, whence $\mathcal{G}_{\Lambda_{A,B}}$ is Hausdorff by [9, Proposition 6.2].

4. TIGHT REPRESENTATIONS OF $\Lambda_{A,B}$

We start this section recalling the definition of a tight representation of a semigroupoid Λ in an inverse semigroup \mathcal{S} with zero.

Definition 4.1 ([9, Definition 15.1]). A representation of the semigroupoid Λ in an inverse semigroup \mathcal{S} with zero is a map $\pi : \Lambda \rightarrow \mathcal{S}$ such that, for every $f, g \in \Lambda$:

- (i) $\pi_f \pi_g = \pi_{fg}$ if $(f, g) \in \Lambda^{(2)}$, and $\pi_f \pi_g = 0$ if $f \perp g$. Moreover, if $q_f^\pi = \pi_f^* \pi_f$ and $p_g^\pi = \pi_g \pi_g^*$ are the associated projections, then
- (ii) $p_f^\pi p_g^\pi = 0$ if $f \perp g$.
- (iii) $q_f^\pi p_g^\pi = p_g^\pi$ if $(f, g) \in \Lambda^{(2)}$.
- (iv) $q_f^\pi p_g^\pi = 0$ if $(f, g) \notin \Lambda^{(2)}$.

If \mathcal{S} is an inverse semigroup of partial isometries on a Hilbert space \mathcal{H} , we may consider the representation as a map $\pi : \Lambda \rightarrow \mathbb{B}(\mathcal{H})$ in which the projections $q_f^\pi = \pi_f^* \pi_f$ and $p_g^\pi = \pi_g \pi_g^*$ commute and the conditions (i-iv) are fulfilled.

Given finite subsets $F, G \subset \Lambda$, we define $\Lambda^{F,G} := \bigcap_{f \in F} \Lambda^f \cap \bigcap_{g \in G} \Lambda \setminus \Lambda^g$.

Definition 4.2 ([9, Definition 15.1]). A representation of the semigroupoid Λ in a Hilbert space \mathcal{H} is said to be tight if for every finite subsets $F, G \subset \Lambda$ and for every finite covering H of $\Lambda^{F,G}$ we have that $\bigvee_{h \in H} p_h = q_{F,G}$, where $q_{F,G} := \prod_{f \in F} q_f \prod_{g \in G} (1 - q_g)$. In particular, if H is a partition, then $\bigvee_{h \in H} p_h = \sum_{h \in H} p_h$.

4.3. In the case of the semigroupoid $\Lambda_{A,B}$, the sets $\Lambda_{A,B}^{F,G}$ have a very precise description. Given $f \in \Lambda_{A,B}$, denote $t(f) = j$ if either $f = h_j^t$ or $f = f g_{i,j,m}$. Then, note that $\bigcap_{f \in F} \Lambda_{A,B}^f \neq$

\emptyset if and only if $\{t(f) \mid f \in F\}$ is a singleton, and in this case $\bigcap_{f \in F} \Lambda_{A,B}^f = \Lambda_{A,B}^{h_{t(f)}}$. On the other hand, $\bigcap_{g \in G} \Lambda_{A,B} \setminus \Lambda_{A,B}^g = \Lambda_{A,B} \setminus \bigcup_{g \in G} \Lambda_{A,B}^g$, so that $\Lambda_{A,B}^{F,G} \neq \emptyset$ if and only if $\{t(f) \mid f \in F\}$ is a singleton and $t(f) \notin \{t(g) \mid g \in G\}$, and in this case $\Lambda_{A,B}^{F,G} = \Lambda_{A,B}^{h_{t(f)}}$. Hence, $q_{F,G} \neq 0$ if and only if $\{t(f) \mid f \in F\}$ is a singleton and $t(f) \notin \{t(g) \mid g \in G\}$, and in this case $q_{F,G} = q_f$.

Lemma 4.4. *Given any tight representation π of $\mathcal{S}(\Lambda_{A,B})$ on a Hilbert space \mathcal{H} , we have that:*

- (1) $q_{h_i}^\pi = p_{h_i}^\pi = q_{g_{j,i,n}}^\pi$ for all $1 \leq i, j \leq N$, all $i \in \Omega_A(j)$ and all $n \in \mathbb{Z}$.
- (2) $q_{g_{t,i,m}}^\pi = \sum_{j \in \Omega_A(i)} \sum_{n=1}^{A_{i,j}} p_{g_{i,j,n}}^\pi$ for all $1 \leq i, t \leq N$, all $i \in \Omega_A(t)$ and all $m \in \mathbb{Z}$.
- (3) $q_{h_i}^\pi \perp q_{h_j}^\pi$ if $i \neq j$.
- (4) $\pi_{h_i} \pi_{g_{i,j,n}} = \pi_{g_{i,j,n+B_{i,j}}}$ and $\pi_{g_{i,j,n}} \pi_{h_j} = \pi_{g_{i,j,n+A_{i,j}}}$ for all $1 \leq i \leq N$, all $j \in \Omega_A(i)$ and all $n \in \mathbb{Z}$.

Proof. Given a tight representation π of $\mathcal{S}(\Lambda_{A,B})$ on a Hilbert space \mathcal{H} , let us compute three particular cases of the identity $\bigvee_{h \in H} p_h = q_{F,G}$ above:

- (1) If $F = \{h_i\}$ and $G = \emptyset$, then the partition $H = \{h_i\}$ give us $q_{h_i}^\pi = p_{h_i}^\pi$.
- (2) If $F = \{g_{j,i,n}\}$ for some $(j,i) \in \Omega_A$ and $1 \leq n \leq A_{j,i}$, and $G = \emptyset$, then the partition $H = \{h_i\}$ give us $q_{g_{j,i,n}}^\pi = p_{h_i}^\pi$.
- (3) If $F = \{h_i\}$ and $G = \emptyset$, then the partition $H = \{g_{i,j,n} \mid j \in \Omega_A(i), 1 \leq n \leq A_{i,j}\}$ give

$$\text{us } q_{h_i}^\pi = \sum_{j \in \Omega_A(i)} \sum_{n=1}^{A_{i,j}} p_{g_{i,j,n}}^\pi.$$

Hence (1) and (2) hold, while (3) and (4) hold by definition of π and of $\Lambda_{A,B}$. \square

In the reverse sense, we have the following result.

Lemma 4.5. *The map $\tau : \Lambda_{A,B} \rightarrow \mathcal{O}_{A,B}$ defined by:*

- (i) $\tau_{h_i} = u_i$ for all $1 \leq i \leq N$.
- (ii) $\tau_{g_{i,j,n}} = s_{i,j,n}$ for all $1 \leq i \leq N$, all $j \in \Omega_A(i)$ and all $n \in \mathbb{Z}$.
- (iii) If $g \in \Lambda_{A,B}$ is written in standard form as $g = g_1 \cdots g_n$, then $\tau_g = \tau_{g_1} \cdots \tau_{g_n}$.

is a tight representation of $\Lambda_{A,B}$ on a separable Hilbert space that respects least common multiples.

Proof. Consider the above defined map $\tau : \Lambda_{A,B} \rightarrow \mathcal{O}_{A,B}$. Since the standard form is unique by Lemma 3.6, and $s_{i,j,n} u_j = s_{i,j,n+A_{i,j}}$ and $u_i s_{i,j,n} = s_{i,j,n+B_{i,j}}$ for all $(i,j) \in \Omega_A$ and $n \in \mathbb{Z}$, τ is a well-defined map. By definition, τ_f is a partial isometry for each $f \in \Lambda_{A,B}$. The projections p_f^τ and q_g^τ are $s_f s_f^*$ and $q_{t(f)}$ respectively. Hence, they commute (see 2.6). Moreover, properties (i-iv) in Definition 4.1 are fulfilled, so that τ is a representation of $\Lambda_{A,B}$ on $\mathcal{O}_{A,B}$.

In order to see that τ is a tight representation, recall that by 4.3, $\Lambda_{A,B}^{F,G} \neq \emptyset$ if and only if $\{t(f) \mid f \in F\}$ is a singleton and $t(f) \notin \{t(g) \mid g \in G\}$, and in this case $\Lambda_{A,B}^{F,G} = \Lambda_{A,B}^{h_{t(f)}}$. Hence, it is enough to show that, whenever H is a finite partition of $\Lambda_{A,B}^{F,G}$, then $q_f^\tau = \sum_{h \in H} p_h^\tau$.

But notice that by Definition 2.3 we have $q_{h_i}^\tau = q_{g_{t,i,m}}^\tau = \sum_{j \in \Omega_A(i)} \sum_{n=1}^{A_{i,j}} p_{g_{i,j,n}}^\tau$. Since any finite

partition is obtained by expansion of the partitions $\{h_i\}$ and $\{g_{i,j,n} \mid j \in \Omega_A(i), 1 \leq n \leq A_{i,j}\}$ (this is the idea of expansion of a rooted tree used in the proof of Proposition 3.14), and these expansion faithfully transfer to the projections p_f^τ and q_g^τ , we obtain the desired result. Also, because of Proposition 3.14 and [9, Proposition 15.5], τ respects least common multiples. \square

Recall that a representation π of a semigroupoid Λ on a Hilbert space is said to be normal if it is tight and respects least common multiples. Then:

- (1) We will denote \mathcal{O}_Λ the C^* -algebra generated by a universal tight representation π of Λ [10, Definition 4.6].
- (2) We will denote $\mathcal{O}_\Lambda^{\text{lcm}}$ the C^* -algebra generated by a universal normal representation π^u of Λ [9, Definition 18.2].

As remarked in [9, Section 18], if Λ satisfies the statement of Proposition 3.14, then by [9, Proposition 15.5] any tight representation is normal, and so $\mathcal{O}_\Lambda \cong \mathcal{O}_\Lambda^{\text{lcm}}$. Now, the above discussion allows us to prove the following result.

Theorem 4.6. *The C^* -algebras $\mathcal{O}_{A,B}$ and $\mathcal{O}_{\Lambda_{A,B}}$ are naturally isomorphic.*

Proof. Because of Lemma 4.4 and the Universal Property of $\mathcal{O}_{A,B}$, there is a (unique) $*$ -homomorphism $\varphi : \mathcal{O}_{A,B} \rightarrow \mathcal{O}_{\Lambda_{A,B}}$ defined by $\varphi(u_i) = \pi_{h_i}$ and $\varphi(s_{i,j,n}) = \pi_{g_{i,j,n}}$. On the other hand, by Lemma 4.5 and the Universal Property of $\mathcal{O}_{\Lambda_{A,B}}$, there exists a (unique) $*$ -homomorphism $\psi : \mathcal{O}_{\Lambda_{A,B}} \rightarrow \mathcal{O}_{A,B}$ defined by $\psi(\pi_{h_i}) = u_i$ and $\psi(g_{i,j,n}) = s_{i,j,n}$. As both maps are mutually inverse, we conclude that $\mathcal{O}_{A,B} \cong \mathcal{O}_{\Lambda_{A,B}}$. \square

As a consequence we obtain the following result

Corollary 4.7. *If $\mathcal{G}_{\Lambda_{A,B}}$ denotes the groupoid of germs associated to the tight part of the spectrum of $E(\mathcal{S}(\Lambda_{A,B}))$, then $\mathcal{O}_{A,B} \cong C^*(\mathcal{G}_{\Lambda_{A,B}})$.*

Proof. By Proposition 3.7, Proposition 3.11 and Lemma 3.13 $\Lambda_{A,B}$ fulfils the requirements of [9, Theorem 18.4]. Hence, the result holds from [9, Theorem 18.4] and Theorem 4.6. \square

Remark 4.8. Notice that the universal tight representation $\pi : \Lambda_{A,B} \rightarrow \mathcal{O}_{\Lambda_{A,B}}$ extends to a representation $\pi : \mathcal{S}(\Lambda_{A,B}) \rightarrow \mathcal{O}_{A,B}$ which is a monoid homomorphism (sending zero to zero). In fact, $\pi(\mathcal{S}(\Lambda_{A,B}))$ is exactly the inverse semigroup $\mathcal{S}^{A,B}$ of partial isometries of $\mathcal{O}_{A,B}$. Moreover, by Lemmas 3.5 and 3.6, π restricts to a monoid isomorphism between the submonoid

$$\Xi = \{(f, \Lambda^f, 1) \mid f \in \Lambda_{A,B}\} \cup \{0\}$$

of $\mathcal{S}(\Lambda_{A,B})$ and the submonoid

$$\Gamma = \{s_{i_1, i_2, n_1} \cdots s_{i_{k-1}, i_k, n_{k-1}} u_{i_k}^{t_k} \mid 1 \leq n_j \leq A_{i_j, i_{j+1}}, t_k \in \mathbb{Z}\} \cup \{u_i^l \mid 1 \leq i \leq N, l \in \mathbb{N}\} \cup \{0\}$$

of $\mathcal{S}^{A,B}$.

Finally, we are ready to prove the following result, which will be relevant to apply our techniques for studying the algebra $\mathcal{O}_{A,B}$. The first part of it appears in [15, Proposition 5.6]. Since the appearance of Brown and Ozawa's book [5], we may give it a much shorter proof.

Theorem 4.9. *Let $N \in \mathbb{N} \cup \{\infty\}$, and let $A \in M_N(\mathbb{Z}^+)$ and $B \in M_N(\mathbb{Z})$ be matrices satisfying Condition (0). Then:*

- (1) $\mathcal{O}_{A,B}$ is nuclear.

- (2) The groupoid $\mathcal{G}_{\Lambda_{A,B}}$ is amenable.
- (3) $C_r^*(\mathcal{G}_{\Lambda_{A,B}}) = C^*(\mathcal{G}_{\Lambda_{A,B}})$.

Proof.

- (1) By [15, Subsection 4.3], $\mathcal{O}_{A,B}$ is the Cuntz-Pimsner algebra $\mathcal{O}_{X_{A,B}}$ of a certain full C^* -correspondence $X_{A,B}$ over the commutative C^* -algebra $\mathcal{A}_N \cong C_0(\{1, 2, \dots, N\} \times \mathbb{T})$. Since \mathcal{A}_N is nuclear, [5, Theorem 4.6.25] implies that the associated Toeplitz-Cuntz-Pimsner algebra $\mathcal{T}_{X_{A,B}}$ is nuclear. Since $\mathcal{O}_{X_{A,B}}$ is a quotient of $\mathcal{T}_{X_{A,B}}$ and nuclearity passes to quotients [5, Theorem 9.4.4], we conclude that $\mathcal{O}_{A,B}$ is nuclear.
- (2) By Corollary 4.7 we have that $\mathcal{O}_{A,B} \cong C^*(\mathcal{G}_{\Lambda_{A,B}})$. Then, by part (1), $C^*(\mathcal{G}_{\Lambda_{A,B}})$ is nuclear. Thus, by [5, Theorem 9.4.4], so is the reduced groupoid C^* -algebra $C_r^*(\mathcal{G}_{\Lambda_{A,B}})$, which is its quotient. Hence, $\mathcal{G}_{\Lambda_{A,B}}$ is amenable by [5, Theorem 5.6.18].
- (3) Since $\mathcal{G}_{\Lambda_{A,B}}$ is amenable by part (2), the result holds by [5, Corollary 5.6.17].

□

5. COMPUTING THE SPACE $\mathcal{G}_{\Lambda_{A,B}}^{(0)}$

In this section we will explicitly compute a suitable picture for $\mathcal{G}_{\Lambda_{A,B}}^{(0)}$ directly associated to $\mathcal{O}_{A,B}$, and we will use it to give an alternative description of $\mathcal{O}_{A,B}$ in terms of an action α of $\mathcal{S}^{A,B}$ on the new picture of $\mathcal{G}_{\Lambda_{A,B}}^{(0)}$.

To this end recall that, according to [9], $\mathcal{G}_{\Lambda_{A,B}}^{(0)}$ is homeomorphic to the space $E(\widehat{\mathcal{S}(\Lambda_{A,B})})_{\text{tight}}$ of tight characters on the semilattice $E(\mathcal{S}(\Lambda_{A,B}))$ of projections of $\mathcal{S}(\Lambda_{A,B})$.

In order to simplify the notation, along this section we will denote $\mathcal{S} := \mathcal{S}(\Lambda_{A,B})$ and $E := E(\mathcal{S}(\Lambda_{A,B}))$. Following the notation in (3.15) and [9, Notations 19.3], we denote:

- (1) $E_p := \{p_f \mid f \in \Lambda_{A,B}\}$.
- (2) $E_q := \{q_{h_i} \mid 1 \leq i \leq N\}$.

Definition 5.1 ([9, Definition 19.5]). Given a filter ξ in E , we will say that ξ is of:

- (1) p -type, if $\xi \subseteq E_p$.
- (2) q -type, if $\xi \subseteq E_q$.
- (3) pq -type, if $\xi \cap E_p$ and $\xi \cap E_q$ are nonempty.

Definition 5.2 ([9, Definition 19.7]). Given a filter ξ in E , we will say that the stem of ξ is $\omega_\xi = \{f \in \Lambda_{A,B} \mid p_f \in \xi\}$.

Definition 5.3 ([9, Definition 19.10]). A path in $\Lambda_{A,B}$ is a subset ω of $\Lambda_{A,B}$ such that:

- (i) If $f \in \omega$ and $g \in \Lambda_{A,B}$ such that $g|f$, then $g \in \omega$;
- (ii) For every $f, g \in \omega$ one has $f \mathbin{\frown} g$, and moreover $\text{lcm}(f, g) \in \omega$.

In particular [9, Proposition 19.9], any stem ω_ξ of a filter ξ is a path, and the correspondence $\xi \mapsto \omega_\xi$ defines a bijection between filters of E (not of q -type) and paths of $\Lambda_{A,B}$ [9, Proposition 19.11].

5.4. Recall that if $f \in \Lambda_{A,B}$, $r(f)$ denotes the (unique) $\Lambda_{A,B}^{h_i} \in \mathcal{Q}$ such that $f \in \Lambda_{A,B}^{h_i}$. Thus, under the above correspondence, [9, Proposition 19.12] implies that the tight-filters ξ (i.e. those whose associate character $\phi_\xi \in \widehat{E}_{\text{tight}}$) are exactly the pq -type filters ξ satisfying:

- (1) For all $f \in \omega_\xi$ and for all $H \subseteq \Lambda_{A,B}^f$ finite cover, there exists $h \in H$ such that $fh \in \omega_\xi$.
- (2) For all finite cover H of $r(\omega_\xi)$ one has $h \in \omega_\xi$ for some $h \in H$.

Proposition 5.5. *Let ξ be a tight-filter of E , and let ω_ξ be the associated stem. Then, there exists a unique infinite sequence $\{i_k\}_{k \geq 1} \subset \{1, 2, \dots, N\}^\mathbb{N}$ and unique elements $1 \leq n_j \leq A_{i_j, i_{j+1}}$ such that the elements in the stem ω_ξ are exactly of two types:*

- (1) $h_{i_1}^t$, for every $t \in \mathbb{N}$.
- (2) $g_{i_1, i_2, n_1} g_{i_2, i_3, n_2} \cdots g_{i_{k-1}, i_k, n_{k-1}} g_{i_k, i_{k+1}, m_k}$, for every $k \in \mathbb{N}$ and every $m_k \in \mathbb{Z}$ such that $m_k \equiv n_k \pmod{A_{i_k, i_{k+1}}}$.

Proof. First, let $\Lambda_{A,B}^{h_{i_1}} = r(\omega_\xi)$. Then, by Proposition 3.14 and property (2) in (5.4):

- (i) Using the partition $H = \{h_i\}$, we have that $h_{i_1} \in \omega_\xi$, and moreover, since ω_ξ is a path, it is the only $h_j \in \omega_\xi$, as otherwise $p_{h_{i_1}} \perp p_{h_j}$ with both projections in ω_ξ will imply that $0 \in \xi$, which is impossible.
- (ii) Given any $t \in \mathbb{Z}$, and using the partition $H_t = \{g_{i_1, j, n+tA_{i_1, j}} \mid j \in \Omega_A(i_1), 1 \leq i \leq A_{i_1, j}\}$, we have that for each $t \in \mathbb{Z}$ there exists at least a $j_t \in \Omega_A(i_1)$ and a $1 \leq n_t \leq A_{i_1, j_t}$ such that $g_{i_1, j_t, n_t+tA_{i_1, j_t}} \in \omega_\xi$. And since ω_ξ is a path, by the same argument as in (i) there exist exactly one $j_t \in \Omega_A(i_1)$ and one $1 \leq n_t \leq A_{i_1, j_t}$ with this property. By the same reason, for $t \neq r \in \mathbb{Z}$, since both $g_{i_1, j_t, n_t+tA_{i_1, j_t}}$ and $g_{i_1, j_r, n_r+rA_{i_1, j_r}}$ are in ω_ξ , we conclude by the argument in the proof of Proposition 3.11 that $j_t = j_r$ and $n_t = n_r$. Thus, we fix $i_2 = j_0$ and $n_1 = n_0$.

On the other hand, using these computations, jointly with by Proposition 3.14 and property (1) in (5.4):

- (i) Using the partition $H = \{h_i\}$, for all $t \in \mathbb{N}$ we have that $h_{i_1}^t \in \omega_\xi$ (which fits with point (i) in the previous computation).
- (ii) Using the partition $H = \{h_i\}$, for all $t \in \mathbb{Z}$, if $g_{i_1, j, n+tA_{i_1, j}} \in \omega_\xi$ then $g_{i_1, j, n+tA_{i_1, j}} h_j \in \omega_\xi$ (which fits with point (ii) in the previous computation).
- (iii) Given any $t \in \mathbb{Z}$, and using the partition $H_t = \{g_{i_1, j, n+tA_{i_1, j}} \mid j \in \Omega_A(i_1), 1 \leq i \leq A_{i_1, j}\}$, there exists a unique $j \in \Omega_A(i_1)$ and a unique $1 \leq m_t \leq A_{i_1, j}$ such that $h_{i_1} g_{i_1, j, m_t+tA_{i_1, j}} \in \omega_\xi$. Since $h_{i_1} g_{i_1, j, m_t+tA_{i_1, j}} = g_{i_1, j, m_t+tA_{i_1, j}+B_{i_1, j}}$, point (ii) in the previous computation shows that $m_t = n_t - B_{i_1, j}$.
- (iv) Fixed the unique element $g_{i_1, i_2, n_1} \in \omega_\xi$ obtained in the above computation, and using the partition $H_t = \{g_{i_1, j, n+tA_{i_1, j}} \mid j \in \Omega_A(i_1), 1 \leq i \leq A_{i_1, j}\}$, we conclude by the same arguments as above that there exist a unique $i_3 \in \{1, 2, \dots, N\}$ and a unique $1 \leq n_2 \leq A_{i_2, i_3}$ such that $g_{i_1, i_2, n_1} g_{i_2, i_3, n_2} \in \omega_\xi$. Moreover, again as above, $g_{i_1, i_2, n_1} g_{i_2, j, m} \in \omega_\xi$ if and only if $j = i_3$ and $m \equiv n_2 \pmod{A_{i_2, i_3}}$.

Recurrence on the last computation give us the desired result. \square

5.6. In particular, to each tight-filter ξ of E we can associate an infinite path

$$\widehat{\omega}_\xi := g_{i_1, i_2, n_1} g_{i_2, i_3, n_2} \cdots$$

in the alphabet $\{g_{i, j, n} \mid 1 \leq i \leq N, j \in \Omega_A(i), 1 \leq n \leq A_{i, j}\}$, such that, for every $k \in \mathbb{N}$, the element $\widehat{\omega}_\xi|_k = g_{i_1, i_2, n_1} g_{i_2, i_3, n_2} \cdots g_{i_k, i_{k+1}, n_k}$ belongs to ω_ξ ; clearly, $\widehat{\omega}_\xi$ is unique. We will denote by X_A the set of infinite paths in the above alphabet, which is a totally disconnected locally

compact Hausdorff space when endowed with the natural product topology. Also notice that, through the representation $\pi : \mathcal{S} \rightarrow \mathcal{O}_{A,B}$, the infinite path $\widehat{\omega}_\xi$ goes to the infinite path $\pi\widehat{\omega}_\xi = s_{i_1,i_2,n_1}s_{i_2,i_3,n_2}\cdots$ of X_A (and this is clearly a bijection).

In the reverse direction we have

Definition 5.7. Let $\gamma = g_{i_1,i_2,n_1}g_{i_2,i_3,n_2}\cdots$ be an infinite path in the alphabet $\{g_{i,j,n} \mid 1 \leq i \leq N, j \in \Omega_A(i), 1 \leq n \leq A_{i,j}\}$. Then define

$$\omega(\gamma) := \{h_{i_1}^t \mid t \in \mathbb{N}\} \cup \{\gamma|_{k-1}g_{i_k,i_{k+1},m_k} \mid k \in \mathbb{N}, m_k \equiv n_k \pmod{A_{i_k,i_{k+1}}}\}.$$

Thus, we have

Proposition 5.8. *The set $\omega(\gamma)$ is a tight path such that $\widehat{\omega(\gamma)} = \gamma$.*

Proof. By definition it is clear that $f, g \in \omega(\gamma)$ implies $f \mathbin{\frown} g$. Also, it is clear that $f \in \omega(\gamma)$ and $g \in \Lambda_{A,B}$ with $g|f$ implies $g \in \omega(\gamma)$. Hence, $\omega(\gamma)$ is a path. Moreover, the filter ξ associated to $\omega(\gamma)$ is of pq -type.

Now, let us prove that ξ is tight. For this we need to check properties (1-2) in (5.4). It is enough to check it for finite partitions containing either h_j or $g_{j,k,m}$. And that holds by definition of $\omega(\gamma)$. The final statement is obvious, so we are done. \square

Remark 5.9. Notice that ω defines a bijection between infinite paths in the alphabet $\{g_{i,j,n} \mid 1 \leq i \leq N, j \in \Omega_A(i), 1 \leq n \leq A_{i,j}\}$ and tight paths on $\Lambda_{A,B}$. Moreover, by Remark 4.8 and (5.6), the bijection becomes a bijection between infinite paths in the alphabet $\{s_{i,j,n} \mid (i,j) \in \Omega_A, 1 \leq n \leq A_{i,j}\}$ and tight paths in the monoid Γ .

As a consequence of Remark 5.9, Proposition 5.8 and (5.6), we obtain the following consequence

Corollary 5.10. *There exists a bijection $\Phi : \widehat{E}_{\text{tight}} \rightarrow X_A$, defined by the rule $\Phi(\phi_\xi) = \widehat{\omega}_\xi$.*

Now, we will prove that X_A is homeomorphic to $\widehat{E}_{\text{tight}}$.

5.11. Recall that, whenever $\xi \subset E$ is a filter, then the associated character ϕ_ξ is given by the rule $\phi_\xi(x) = 1$ if $x \in \xi$, and $\phi_\xi(x) = 0$ otherwise. Hence, the topology on $\widehat{E}_{\text{tight}}$ is endowed by the inclusion $\widehat{E}_{\text{tight}} \subset \{0,1\}^E$. In particular, the basic open neighborhoods of ϕ_η in $\widehat{E}_{\text{tight}}$ are of the form

$$W = \{\phi_\xi \in \widehat{E}_{\text{tight}} \mid p_{t_1}, \dots, p_{t_k} \in \xi \text{ and } p_{s_1}, \dots, p_{s_l} \notin \xi\}$$

for some $t_1, \dots, t_k, s_1, \dots, s_l \in \mathcal{S}^{A,B}$.

On the other hand, the topology of X_A is defined by the following fundamental system of open neighborhoods: for each $n \in \mathbb{N}$ and each $\gamma \in X_A$, the basic open (in fact clopen) sets are of the form

$$W_n^\gamma = \{\tau \in X_A \mid \gamma|_n = \tau|_n\}.$$

So, we have

Proposition 5.12. *The map $\Phi : \widehat{E}_{\text{tight}} \rightarrow X_A$ is an homeomorphism of topological spaces.*

Proof. First notice that, given an element $x \in \mathcal{S}^{A,B}$, it is either a projection in E , u_i^t for some $t \in \mathbb{N}$, or can be written as $x = s_I u_{s(I)}^t s_J^*$ for I, J multinindices of minimum length and $t \in \mathbb{Z}$. If n denote the length of J , then given any $\gamma \in X_A$ the product $x \cdot \gamma|_n$ can be either zero, or $s_I u_{s(I)}^t$ if $\gamma = s_J \gamma'$. Hence, we can define the operation $x \cdot \gamma$ by zero is so is $x \cdot \gamma|_n$, or by $s_I u_{s(I)}^t \gamma'$ otherwise. Obviously, when x is a projection, $x \cdot \gamma \neq 0$ if and only if $x \cdot \gamma = \gamma$. If $\xi \subset E$, then $x \in \xi$ if and only if $x \cdot \widehat{\omega}_\xi \neq 0$ if and only if $\phi_\xi(x) = 1$.

By using the identification given by the map ω defined before and Remark 5.9, we can endow a topology on X_A through the inclusion $X_A \subset \{0, 1\}^\Gamma$. In particular, the basic open neighborhoods of η in X_A are of the form

$$W = \{\gamma \in X_A \mid t_1, \dots, t_k \in \omega(\gamma) \text{ and } s_1, \dots, s_l \notin \omega(\gamma)\}$$

for some $t_1, \dots, t_k, s_1, \dots, s_l \in \Gamma$. By the previous paragraph, we can express these open sets as

$$W = \{\gamma \in X_A \mid p_{t_i} \cdot \gamma = \gamma \text{ and } p_{s_j} \cdot \gamma = 0\}$$

for $t_1, \dots, t_k, s_1, \dots, s_l \in \Gamma$. Also, for $\gamma \in X_A$ and $n \in \mathbb{N}$, we can express

$$W_n^\gamma = \{\tau \in X_A \mid p_{(\gamma|_n)} \cdot \tau = \tau\}.$$

In particular, defining $t_1 = \gamma|_n$ and taking any $s_1 \in \Gamma$ such that $s_1^* \cdot \gamma|_n = 0$, we have that W_n^γ is an open neighborhood of γ of the form $W = \{\gamma \in X_A \mid t_1 \in \omega(\gamma) \text{ and } s_1 \notin \omega(\gamma)\}$.

Now, fix $W = \{\gamma \in X_A \mid t_1, \dots, t_k \in \omega(\gamma) \text{ and } s_1, \dots, s_l \notin \omega(\gamma)\}$ an open neighborhood of $\eta \in X_A$; in particular $p_{t_i} \cdot \eta = \eta$ and $p_{s_j} \cdot \eta = 0$. Set $n = \max\{|t_1|, \dots, |t_k|, |s_1|, \dots, |s_l|\}$; we will assume that the length of u_i is zero. Thus, if $\omega \in W_n^\eta$, since $\omega = \eta|_n \cdot \widehat{\omega}$, we have that $p_{t_i} \cdot \omega = \omega$ and $p_{s_j} \cdot \omega = 0$. Hence, $W_n^\eta \subset W$, whence, both topologies coincide on X_A .

Thus, for any $W = \{\phi_\xi \in \widehat{E}_{\text{tight}} \mid p_{t_1}, \dots, p_{t_k} \in \xi \text{ and } p_{s_1}, \dots, p_{s_l} \notin \xi\}$ we have that $\Phi(W) = \{\gamma \in X_A \mid t_1, \dots, t_k \in \omega(\gamma) \text{ and } s_1, \dots, s_l \notin \omega(\gamma)\}$. Hence, $\Phi : \widehat{E}_{\text{tight}} \rightarrow X_A$ is a continuous open bijective map, and thus an homeomorphism. \square

The final step is to describe the action $\alpha : \mathcal{S}^{A,B} \rightarrow \widehat{E}_{\text{tight}}$ when considered as an action $\alpha : \mathcal{S}^{A,B} \rightarrow X_A$ via the homeomorphism Φ .

5.13 ([9, Section 10]). Let $s \in \mathcal{S}^{A,B}$, and consider two open sets of $\widehat{E}_{\text{tight}}$:

$$D_{s^*s} = \{\phi \in \widehat{E}_{\text{tight}} \mid \phi(s^*s) = 1\}$$

and

$$D_{ss^*} = \{\phi \in \widehat{E}_{\text{tight}} \mid \phi(ss^*) = 1\}.$$

Then, the action $\alpha : \mathcal{S}^{A,B} \rightarrow \widehat{E}_{\text{tight}}$ is defined by (partial) homeomorphisms

$$\Theta_s : D_{s^*s} \longrightarrow D_{ss^*}$$

under the rule $\Theta_s(\phi)(x) = \phi(s^*xs)$ [9, Proposition 10.3].

Now, we are ready to prove the main result of this section

Theorem 5.14. *Under the identification provided by the homeomorphism $\Phi : \widehat{E}_{\text{tight}} \rightarrow X_A$ of Proposition 5.12, the usual action $\alpha : \mathcal{S}^{A,B} \rightarrow \widehat{E}_{\text{tight}}$ becomes an action $\alpha : \mathcal{S}^{A,B} \rightarrow X_A$ given by multiplication of elements of X_A on the left by elements of $\mathcal{S}^{A,B}$.*

Proof. Fix an element $s \in \mathcal{S}^{A,B}$. If we apply the homeomorphism Φ to the open sets D_{s^*s} and D_{ss^*} , then we obtain open sets

$$X_{s^*s} = \{\gamma \in X_A \mid s^*s\gamma = \gamma\}$$

and

$$X_{ss^*} = \{\gamma \in X_A \mid ss^*\gamma = \gamma\} = \{\gamma \in X_A \mid \gamma = s\gamma'\}.$$

Moreover, the maps

$$\alpha_s : X_{s^*s} \longrightarrow X_{ss^*}$$

given by $\alpha_s(\gamma) = s \cdot \gamma$ are well-defined homeomorphisms, which correspond to Θ under the identification given by Φ . Hence, the usual action α of $\mathcal{S}^{A,B}$ on $\widehat{E}_{\text{tight}}$ becomes the natural action of $\mathcal{S}^{A,B}$ on X_A given by multiplication on the left, as desired. \square

Because of Proposition 5.12 and Theorem 5.14, we can apply [9, Section 4] to give a useful picture of $\mathcal{G}_{\Lambda_{A,B}}$. To be concrete

5.15. Consider the set

$$\Omega = \{(s, x) \in \mathcal{S}^{A,B} \times X_A \mid x \in X_{s^*s}\}.$$

Given $(s, x), (t, y) \in \Omega$, we say that $(s, x) \sim (t, y)$ if $x = y$, and there exists $e \in E(\mathcal{S}^{A,B})$ such that $x \in X_e$ and $se = te$. By replacing e by es^*st^*t , we can assume that $e \leq s^*s, t^*t$. Moreover, since $x = y$, we conclude that either $s^*s \leq t^*t$ or $t^*t \leq s^*s$.

Hence, $\mathcal{G}_{\Lambda_{A,B}}$ is isomorphic to the groupoid of germs $\mathcal{G} := \Omega / \sim$, where $\mathcal{G}^{(0)}$ is identified with X_A by the rule $[e, x] \leftrightarrow x$ for any $x \in X_A$ and any $e \in E(\mathcal{S}^{A,B})$ such that $x \in X_e$. Under this identification, the maps $d, r : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$ are given by $d[s, x] = x$ and $r[s, x] = s \cdot x$. The composition rule is $[s, x] \cdot [t, y] = [st, y]$ whenever $x = t \cdot y$, while $[s, x]^{-1} = [s^*, s \cdot x]$. Recall that the sets $\Omega(s, U) := \{(s, \omega) \mid s \in \mathcal{S}^{A,B}, \omega \in U\}$, where $U \subset X_{s^*s}$ is an open set of X_A , give us a basis of slices (open compact sets) of \mathcal{G} .

An interesting consequence is the following

Corollary 5.16. *The groupoid $\mathcal{G}_{\Lambda_{A,B}}$ is étale with second countable unit space.*

Proof. By Proposition 5.12, $\mathcal{G}_{\Lambda_{A,B}}^{(0)}$ is homeomorphic to X_A , which is a (locally) compact Hausdorff space. Also, X_A is second countable. Finally, the picture of $\mathcal{G}_{\Lambda_{A,B}}$ given in (5.15) proves that the map $r : \mathcal{G}_{\Lambda_{A,B}} \rightarrow \mathcal{G}_{\Lambda_{A,B}}^{(0)}$ is a local homeomorphism. \square

So, we have the following result

Proposition 5.17. *The C^* -algebra $C^*(\mathcal{G}_{\Lambda_{A,B}})$ is isomorphic to the inverse semigroup crossed product $C_0(\mathcal{G}_{\Lambda_{A,B}}^{(0)}) \rtimes_{\alpha} \mathcal{S}^{A,B}$.*

Proof. By Corollary 5.16, $\mathcal{G}_{\Lambda_{A,B}}$ is an étale groupoid with second countable unit space. Clearly, $\mathcal{S}^{A,B}$ is countable. Hence, the result holds from [9, Proposition 9.9]. \square

Thus, we obtain the following picture of $\mathcal{O}_{A,B}$

Corollary 5.18. *The C^* -algebra $\mathcal{O}_{A,B}$ is isomorphic to the inverse semigroup crossed product $C_0(X_A) \rtimes_{\alpha} \mathcal{S}^{A,B}$.*

Proof. The result is a direct consequence of Corollary 4.7, Proposition 5.17, Proposition 5.12 and Theorem 5.14. \square

Notice that, when $B = (0)$, Corollary 5.18 recover the picture of the Exel-Laca algebra \mathcal{O}_A given in [12].

6. INVARIANT SETS AND MINIMALITY

In this section we will study the invariant open subsets of X_A by the action of $\mathcal{S}^{A,B}$, and we characterize the minimality of $\mathcal{G}_{\Lambda_{A,B}}$.

Definition 6.1. If S is a semigroup, and τ is an action by (partial) homeomorphisms on a topological space X , then:

- (1) We say that $x, y \in X$ are of equivalent trajectory under τ , denoted $x \sim_\tau y$ if there exist $s, t \in S$ such that $\tau_s(x)$ and $\tau_t(y)$ are defined and coincide.
- (2) We say that a subset W of X is invariant if for every $y \in W$ and $x \in X$, $x \sim_\tau y$ implies that $x \in W$.
- (3) We say that X is irreducible if it has no proper open invariant subsets.

In our case, we have the following

Lemma 6.2. *Given a subset W of X_A , the following are equivalent:*

- (1) *W is invariant.*
- (2) *For every $t \in \mathcal{S}^{A,B}$ we have that $\alpha_t(W \cap X_{t^*t}) \subseteq W$.*

Proof.

(1) \Rightarrow (2). Let $y \in W$, and let $t \in \mathcal{S}^{A,B}$ with $x = \alpha_t(y) = t \cdot y$. Now, set $s = tt^*$, and notice that $s \cdot x$ is defined, and moreover $s \cdot x = tt^* \cdot x = tt^*t \cdot y = t \cdot y$. Hence, $\alpha_s(x) = \alpha_t(y)$, and thus $x \in W$.

(2) \Rightarrow (1). Let $y \in W$ and let $x \in X$ such that $x \sim_\tau y$. Hence, there exists $s, t \in \mathcal{S}^{A,B}$ such that $s \cdot x = t \cdot y$. Then, we have $x = s^*t \cdot y = \alpha_{s^*t}(y)$, and thus $x \in W$. \square

Remark 6.3. By Lemma 6.2 and (5.15), minimality of $\mathcal{G}_{\Lambda_{A,B}}$ (see e.g. [1] for a definition) faithfully translates to irreducibility of X_A under the action of $\mathcal{S}^{A,B}$.

Definition 6.4. Given a subset C of X_A , we define the orbit of C to be the set

$$\text{Orb}(C) = \bigcup_{t \in \mathcal{S}^{A,B}} \alpha_t(C \cup X_{t^*t}).$$

Notice that $\text{Orb}(C)$ is the smallest invariant subset of X_A containing C .

For each $\gamma \in X_A$ and each $n \in \mathbb{N}$ we define

$$\Delta_n^\gamma = \text{Orb}(W_n^\gamma).$$

Hence, $\{\Delta_n^\gamma \mid \gamma \in X_A, n \in \mathbb{N}\}$ is a fundamental system of invariant open neighborhoods of X_A . In particular, if $x \in \Sigma := \{s_{i,j,n} \mid (i,j) \in \Omega_A, 1 \leq n \leq A_{i,j}\} \cup \{u_i \mid 1 \leq i \leq N\}$, since $\alpha_x(X_{x^*x}) = X_{xx^*}$ and $\alpha_{x^*}(X_{xx^*}) = X_{x^*x}$, then both open sets have the same orbit, that we will denote Δ_x . Given $x = u_i$, note that $u_i^*u_i = u_iu_i^* = p_i$, so that $\alpha_x(X_{xx^*}) = X_{xx^*} = X_{x^*x}$, and $p_i = s_{j,i,n}^*s_{j,i,n}$, so that the action of u_i do not give extra information when studying invariance of open subsets of X_A . Thus, we can restrict our attention to the action of elements in $\Phi = \{s_{i,j,n} \mid (i,j) \in \Omega_A, 1 \leq n \leq A_{i,j}\}$

Now, we have the following

Lemma 6.5. *If $U \subset X_A$ is a nonempty invariant open set, then there exists $x \in \Phi$ such that $\Delta_x \subseteq U$.*

Proof. By (5.11) there exists $n \in \mathbb{N}$ and $\gamma \in X_A$ such that $W_n^\gamma \subseteq U$. Now, define $t := \gamma|_n$ and $x := t_n$. Without loss of generality we can assume that $n > 1$, so that $t \in \mathcal{S}^A$, and $x \in \Phi$.

If $\eta \in W_n^\gamma$, then $\eta = t \cdot \hat{\eta}$, so that $\hat{\eta} = t^* \cdot \eta$. Since $t^*t = x^*x$, we conclude that $\alpha_{t^*}(\eta) = \hat{\eta} \in X_{x^*x}$. Conversely, if $\eta \in X_{x^*x}$, then $\eta = x^*x \cdot \eta = t^*t \cdot \eta$, so that $\alpha_t(\eta) = t \cdot \eta \in W_n^\gamma$. Hence, W_n^γ and X_{x^*x} are in the same orbit, so that

$$\Delta_x = \Delta_n^\gamma \subseteq \text{Orb}(U) = U,$$

as desired. \square

Lemma 6.6. *If $x, y \in \Phi$ and there exists $t \in \mathcal{S}^A$ starting at x and ending at y , then $\Delta_y \subseteq \Delta_x$.*

Proof. Let $t \in \mathcal{S}^A$ such that $x = t_1$ and $y = t_{|t|}$. Now, if $\eta \in X_{y^*y}$, then $\eta = y^*y \cdot \eta$. Since $y^*y = t^*t$, we have that $\eta \in X_{t^*t}$, and thus $\alpha_t(\eta) = t \cdot \eta \in X_{xx^*}$. Taking orbits, we conclude that $\Delta_y \subseteq \Delta_x$, as desired. \square

Given $x \in \Phi$, we denote $T_x = \{y \in \Phi \mid t(x) = d(y)\}$. Since A is row-finite, $|T_x| < \infty$ for every $x \in \Phi$. Then

Corollary 6.7. *For any $x \in \Phi$, $\Delta_x = \bigcup_{y \in T_x} \Delta_y$.*

Proof. By Lemma 6.6, $\bigcup_{y \in T_x} \Delta_y \subseteq \Delta_x$. Now, let $\eta \in X_{x^*x}$. Since $\eta = x^*x \cdot \eta$, we have that $\eta = y \cdot \hat{\eta}$ for some $y \in T_x$. Thus, $\eta \in X_{yy^*}$. Taking orbits, we conclude that $\Delta_x \subseteq \bigcup_{y \in T_x} \Delta_y$, as desired. \square

As immediate consequence, we get the following result

Corollary 6.8. *For the action α of $\mathcal{S}^{A,B}$ on X_A , the following are equivalent:*

- (1) *The matrix A is irreducible.*
- (2) *The space X_A is irreducible.*
- (3) *The groupoid $\mathcal{G}_{\Lambda_{A,B}}$ is minimal.*

Proof.

- (1) \Rightarrow (2) It is a direct consequence of Lemma 6.5 and Corollary 6.7.
- (2) \Rightarrow (1) If A is not irreducible, there exist $x, y \in \Phi$ such that no element of \mathcal{S}^A starts at x and ends at y . Hence, Δ_x turns out to be a proper open invariant subset, so we are done.
- (2) \Leftrightarrow (3) It is a consequence of Remark 6.3. \square

7. ESSENTIALLY PRINCIPAL GROUPOIDS AND TOPOLOGICALLY FREE ACTIONS

In this section we will connect the notion of being topological free for X_A with that of being essentially principal for $\mathcal{G}_{\Lambda_{A,B}}$. Then, we will analyse fixed points and topological freeness of X_A . Given a subset C of a topological space X , we denote its interior by $\text{Int}(C)$. Along this section we will assume that matrix B satisfies Condition (E), so that $\mathcal{S}^{A,B}$ is a E^* -unitary semigroup (i.e. a 0- E -semigroup).

Definition 7.1. Let S be an inverse semigroup, and let τ be an action of S on a topological space X .

- (1) Given $s \in S$ and $x \in X_{s^*s}$, we say x is a fixed point for s if $\tau_s(x) = x$.
- (2) If x is a fixed point for s , we say that x is a trivial fixed point if there exists $e \in E(S)$, such that $x \in X_e$, and $se = e$.
- (3) We say that the action is free if every fixed point for every s in S is trivial.
- (4) We say that the action is topologically free if, for every $s \in S$, the interior of the set of fixed points for s consists of trivial fixed points.

Remark 7.2.

- (1) It is easy to see that if e is an idempotent elements in S , then every x in X_e is a trivial fixed point for e .
- (2) On the other hand, when S is E^* -unitary, it is not hard to show that for $s \in S \setminus E(S)$, every fixed point of s is nontrivial.

Thus, for the special case of E^* -unitary inverse semigroups we have:

Theorem 7.3. *Let τ be an action of an E^* -unitary inverse semigroup S on a topological space X . Then:*

- (1) τ is free if and only if, for every $s \in S \setminus E(S)$, the set of fixed points for s is empty.
- (2) τ is topologically free if and only if, for every $s \in S \setminus E(S)$, the set of fixed points for s has empty interior.

We will connect topological freeness with a suitable notion for groupoids, which will help us to characterize simplicity for our algebras.

Definition 7.4. Let \mathcal{G} be a locally compact, Hausdorff, étale groupoid. Then:

- (1) For any $x \in \mathcal{G}^{(0)}$, the isotropy group at x is
$$\mathcal{G}(x) = \{\gamma \in \mathcal{G} \mid d(\gamma) = t(\gamma) = x\}.$$
- (2) \mathcal{G} is principal if for every $x \in \mathcal{G}^{(0)}$ we have $\mathcal{G}(x) = \{x\}$.
- (3) \mathcal{G} is essentially principal if the interior of the isotropy group bundle

$$\mathcal{G}' = \{\gamma \in \mathcal{G} : d(\gamma) = t(\gamma)\}$$

is contained in $\mathcal{G}^{(0)}$.

Notice that, if \mathcal{G} is second countable, Hausdorff, étale, and the unit space has the Baire property, then \mathcal{G} is essentially principal if and only if for every $x \in \mathcal{G}^{(0)}$ the set $\{x \in \mathcal{G}^{(0)} \mid \mathcal{G}(x) = \{x\}\}$ is dense in $\mathcal{G}^{(0)}$ [23, Proposition 3.1].

Now, we have the following result, connecting both notions.

Theorem 7.5. *Let S be an E^* -unitary inverse semigroup, let τ be an action of S on a locally compact, Hausdorff space X , and let \mathcal{G} be the corresponding groupoid of germs. Then \mathcal{G} is essentially principal if and only if τ is topologically free.*

Proof. As seen above, recall that τ is topologically free if and only if,

- (\star) for every $s \in S \setminus E(S)$, the interior of the set of fixed points for s is empty.

Suppose that \mathcal{G} is essentially principal, and let $s \in S \setminus E(S)$. By way of contradiction suppose that x is an interior fixed point for s , so there exists an open subset U of X_{s^*s} , consisting of fixed points for s , such that $x \in U$. It follows that $\Omega(s, U)$ is contained in \mathcal{G}' , and hence also in $\mathcal{G}^{(0)}$, by hypothesis. In particular $[s, x] \in \mathcal{G}^{(0)}$, from where we deduce that $[s, x] = [e, x]$, for some idempotent e in S .

The condition for equality of germs gives an idempotent f in S , such that $x \in X_f$, and $sf = ef$. Since $x \in X_e \cap X_f = X_{ef}$, it is clear that ef is nonzero. We conclude that s dominates the nonzero idempotent ef , so s itself is idempotent, a contradiction.

Conversely, assume that τ is topologically free, and let γ lie in the interior of \mathcal{G}' . One may then choose $s \in S$, and an open set $U \subseteq X_{s^*s}$, such that

$$\gamma \in \Omega(s, U) \subseteq \mathcal{G}'.$$

So U is formed by fixed points for s , and hence U is contained in the interior of the set of fixed points for s . By (\star) we may rule out the possibility that $s \in S \setminus E(S)$, in turn concluding that $s \in E(S)$, and hence that $\gamma \in \mathcal{G}^{(0)}$. This shows that \mathcal{G} is essentially principal. \square

Remark 7.6. Recall that, if the matrix B satisfies Condition (E), then $\mathcal{S}^{A,B}$ is E^* -unitary, and thus Theorem 7.5 applies. This is the reason why we will require this property as hypothesis in the sequel.

Since the property of being essentially principal plays a major role in deciding the simplicity of $C_r^*(\mathcal{G}_{\Lambda_{A,B}})$ (see e.g. [4, Theorem 5.1]), Theorem 7.5 means that analysing fixed points and topological freeness for X_A is an unavoidable step towards the characterisation of simplicity for $\mathcal{O}_{A,B}$ in this context.

We will start by studying the action α restricted to the submonoid \mathcal{S}^A . Then, we have the following result

Lemma 7.7. *Let $t \in \mathcal{S}^A \setminus E(\mathcal{S}^{A,B})$, and let $\omega \in X_{t^*t}$ such that $\alpha_t(\omega) = \omega$. Then:*

- (1) *There exists finite multiindices $I \neq K$ such that $\omega = s_I s_K s_K \cdots$.*
- (2) *The set $S_t = \{\eta \in X_A \mid \alpha_t(\eta) = \eta\} = \{\omega\}$.*

Proof. Such an element t must be either s_I, s_J^* or $s_I s_J^*$ for finite multiindices $I \neq J$. Let us distinguish cases:

- (i) If $t = s_I$, then $\omega = s_I \cdot \omega$. Hence, by recurrence, $\omega = s_I s_I s_I \cdots$.
- (ii) If $t = s_J^*$, then $\omega = s_J^* \omega$ forces $\omega = s_J \hat{\omega}$. Hence, $\hat{\omega} = s_J^* s_J \hat{\omega} = s_J^* \omega = \omega = s_J \hat{\omega}$. Thus, $\omega = s_J s_J s_J \cdots$.
- (iii) If $t = s_I s_J^*$, then $s_I s_J^* \cdot \omega = \omega = s_I \cdot \hat{\omega}$, so that $s_J^* \cdot \omega = \hat{\omega}$. But $\omega = s_I s_J^* \cdot \omega = (s_I s_J^*)^2 \cdot \omega$ implies that $s_J^* s_I \neq 0$. Thus, either $J = IK$ or $I = JK$ for $K \neq \emptyset$. In the first case, $\omega = s_I s_K^* s_I^* \cdot \omega$ so that $\omega = s_I s_K \cdot \hat{\omega}$. Then,

$$s_I \cdot \hat{\omega} = \omega = s_I s_K^* s_I^* \cdot \omega = s_I s_K^* \cdot \hat{\omega},$$

so that $\hat{\omega} = s_K \cdot \hat{\omega}$. Thus, $\omega = s_I s_K s_K \cdots$. In the second case, an analogue argument works, and we get $\omega = s_J s_K s_K \cdots$.

So we are done. \square

Recall that, given a matrix $A \in M_N(\mathbb{Z}^+)$, we define the associated directed graph E_A by taking as vertices the set $\{1, 2, \dots, N\}$ and as edges the set $\{s_{i,j,n} \mid (i, j) \in \Omega_A, 1 \leq n \leq A_{i,j}\}$. We say that E_A :

- (1) Satisfies Condition (L) if every cycle has an exit.
- (2) Satisfies Condition (K) if every vertex in a cycle is a basis for at least two different cycles.

Then, we have the following consequence

Proposition 7.8 (c.f. [12, Proposition 12.2]). *The restriction of the action α to $\mathcal{S}^A \setminus E(\mathcal{S}^{A,B})$ is topologically free if and only if the graph E_A satisfies Condition (L).*

Proof. Notice that E_A fails Condition (L) if and only if there exists a finite sequence i_1, i_2, \dots, i_k such that $(i_j, i_{j+1}), (i_k, i_1) \in \Omega_A$ and $A_{i_k, i_1} = A_{i_j, i_{j+1}} = 1$ for all j .

Let $t \in \mathcal{S}^A \setminus E(\mathcal{S}^{A,B})$ be such that $S_t \neq \emptyset$. Then, by Lemma 7.7, there exist finite multiindices $I \neq K$ such that $S_t = \{\omega\}$ for the infinite path $\omega = s_I s_K s_K \dots$. Hence, $\text{Int}(S_t) = \emptyset$ if and only if ω is not an isolated point.

Suppose that E_A satisfies Condition (L). Then, either there exists $l \neq i_{|I|+1}$ such that either $(i_{|I|+2}, l) \in \Omega_A$ or $A_{i_{|I|+1}, i_{|I|+2}} > 1$, so that for a suitable m we have $s_{i_{|I|+1}, l, m}$ is different of the initial section of s_K , $s_{i_{|I|+1}, i_{|I|+2}, n_1}$. Thus, given any $n \in \mathbb{N}$, there exists $r \in \mathbb{N}$ such that $|I|r \cdot |K| > n$. Hence, we can construct a path $\omega \neq \gamma = s_I s_K \dots s_K s_{i_1, l, m} \hat{\gamma}$ such that $\gamma \in W_n^\omega$, whence $S_t \neq W_n^\omega$. Then, ω is not isolated.

Conversely, if E_A fails Condition (L), then there exists a terminal circuit s_I . Let $t = s_I$ and $\omega = s_I s_I s_I \dots$. So, $\omega \in X_{t^*}$ and $\alpha_t(\omega) = t \cdot \omega = \omega$, whence $S_t = \{\omega\}$. Now, consider the open neighborhood $V = \{\eta \in X_A \mid p_t \cdot \eta = \eta\}$. Certainly, $\omega \in V$. Now, if $\gamma \in V$, the identity $\gamma = p_t \cdot \gamma$ implies that $\gamma = t \cdot \hat{\gamma}$. Since t is a terminal circuit, we have that $\gamma = t t t \dots = s_I s_I s_I \dots = \omega$. Thus, $V = S_t$, so that $\text{Int}(S_t) \neq \emptyset$. \square

Moreover

Proposition 7.9 (c.f. [12, Proposition 12.4]). *Let $t = s_I s_J^* \in \mathcal{S}^A \setminus E(\mathcal{S}^{A,B})$ and let $\omega = s_K s_L s_L \dots$ a fixed point for α_t . Then, ω is isolated in $\text{Orb}(\omega)$ if and only if the cycle s_L is transitory in E_A (i.e. no exit has a return path).*

Proof. Suppose that s_L is transitory, and consider the open neighborhood

$$V = \{\eta \in X_A \mid p_{s_K s_L} \cdot \eta = \eta\}.$$

Let $\eta \in V \cap \text{Orb}(\omega)$, so that there exists $t \in \mathcal{S}^A$ such that $\eta = t \cdot \omega = t s_K s_L s_L \dots$. But since

$$\eta = t \cdot \omega = t s_K s_L s_L \dots = p_{s_K s_L} \cdot \eta = p_{s_K s_L} \cdot t s_K s_L s_L \dots$$

and s_L is transitory, we conclude that $\eta = \omega$. Thus, $V \cap \text{Orb}(\omega) = \{\omega\}$, whence ω is isolated in $\text{Orb}(\omega)$.

Conversely, suppose that ω is isolated in $\text{Orb}(\omega)$. Then, there exists an open neighborhood U of ω such that $\{\omega\} = U \cap \text{Orb}(\omega)$. Hence, there exists $n \in \mathbb{N}$ such that $V = \{\eta \in X_A \mid p_{s_K s_L^n} \cdot \eta = \eta\} \subseteq U$, and thus $\{\omega\} = V \cap \text{Orb}(\omega)$. If s_L is not transitory, there exists a multiindex $T \neq L$ such that $\omega \neq \gamma = s_K s_L^n s_T s_L s_L \dots \in X_A$. Obviously, $\gamma \in V$. Take $t = s_K s_L^n s_T s_K^* \in \mathcal{S}^A$, and notice that $\alpha_t(\omega) = t \cdot \omega = \gamma$. Hence, $\gamma \in \text{Orb}(\omega)$, whence $\gamma \in V \cap \text{Orb}(\omega)$, and thus ω is not isolated. \square

Consequently, the same argument used to prove [12, Proposition 12.3] applies, and we conclude

Proposition 7.10. *The restriction of the action α to $\mathcal{S}^A \setminus E(\mathcal{S}^{A,B})$ is topologically free on every closed invariant subset of X_A if and only if E_A satisfies Condition (K).*

Now, we will study the fixed points for the action α_t , when $t \in \mathcal{S}^{A,B} \setminus \mathcal{S}^A$, i.e. t either equal u_i^m or $s_I u_{t(I)}^m$ for a finite multiindex I and $m \in \mathbb{Z}$.

Remark 7.11. Given $1 \leq i \leq N$, if $B_{i,j} = 0$ for every $j \in \Omega_A(i)$ (equivalently, if the i -th row of B is identically zero), then by Definition 2.3(i) we have $u_i s_{i,j,n} = s_{i,j,n}$ for every $j \in \Omega_A(i)$ and every $1 \leq n \leq A_{i,j}$. Hence, Definition 2.3 (i) and (iii) imply that $q_i = u_i q_i = u_i$, whence $u_i, u_i^* \in E(\mathcal{S}^{A,B})$. In particular, if $B = (0)$, we conclude that we only need to care about the restriction of α to $\mathcal{S}^A \setminus E(\mathcal{S}^{A,B})$; this is coherent with the fact that $\mathcal{O}_{A,(0)} \cong \mathcal{O}_A$.

Thus, we will assume that $B \neq (0)$ and, when studying the action of u_i^l on X_A for a concrete $1 \leq i \leq N$, that there exists at least one $j \in \Omega_A(i)$ such that $B_{i,j} \neq 0$.

Lemma 7.12. *Given an element $\omega = s_{i_1,i_2,n_1} s_{i_2,i_3,n_2} \cdots s_{i_k,i_{k+1},n_k} \cdots$ of X_A , the following are equivalent:*

- (1) ω is fixed under the action of $u_{i_1}^l$ ($l \in \mathbb{Z}$).
- (2) For every $j \geq 1$ the element $K_j := l \cdot \prod_{t=1}^j \frac{B_{i_t,i_{t+1}}}{A_{i_t,i_{t+1}}}$ belongs to \mathbb{Z} .

Proof. By Definition 2.3(i), $\omega = u_{i_1}^l \cdot \omega$ if and only if there exists a sequence $(K_j)_{j \geq 0} \subseteq \mathbb{Z}$ such that:

- (i) $K_0 = l$.
- (ii) For every $j \geq 1$, $n_{j-1} + K_{j-1} B_{i_j,i_{j+1}} = n_{j-1} + K_j A_{i_j,i_{j+1}}$.

Notice that (ii) is equivalent to ask $K_{j-1} B_{i_j,i_{j+1}} = K_j A_{i_j,i_{j+1}}$ for every $j \geq 1$.

Now, for $j = 1$ we have $l \cdot B_{i_1,i_2} = K_1 A_{i_1,i_2}$, so that $K_1 = l \cdot \frac{B_{i_1,i_2}}{A_{i_1,i_2}}$. Now, suppose that for $1 \leq r \leq j-1$ we have proved that $K_j := l \cdot \prod_{t=1}^r \frac{B_{i_t,i_{t+1}}}{A_{i_t,i_{t+1}}}$. Hence

$$K_j A_{i_j,i_{j+1}} = K_{j-1} B_{i_j,i_{j+1}} = l \cdot \prod_{t=1}^r \frac{B_{i_t,i_{t+1}}}{A_{i_t,i_{t+1}}} \cdot B_{i_j,i_{j+1}},$$

so that $K_j = l \cdot \prod_{t=1}^j \frac{B_{i_t,i_{t+1}}}{A_{i_t,i_{t+1}}}$. This completes the proof. \square

Let us show some characteristic examples of fixed elements.

Example 7.13.

- (1) Suppose that there exists $k \in \mathbb{N}$ such that $B_{i_k,i_{k+1}} = 0$. Thus, given

$$\omega = s_{i_1,i_2,n_1} s_{i_2,i_3,n_2} \cdots s_{i_k,i_{k+1},n_k} s_{i_{k+1},j,m} \cdots,$$

if we take $l = \prod_{t=1}^k A_{i_t,i_{t+1}}$ then we have that ω is fixed by $u_{i_1}^l$ for any choice of sequences starting on $(i_{k+1}, j) \in \Omega_A$. In particular, for any $n \geq k+1$, the open set W_n^ω is contained in $\{\omega \in X_A \mid u_i^l \cdot \omega = \omega\}$, and thus the set of fixed points has nonempty interior.

- (2) If $A \in M_N(\{0, 1\})$, then for any $\omega \in X_{q_i}$ we have that $u_i \cdot \omega = \omega$, so that for any $l \in \mathbb{Z}$ the set $\{\omega \in X_A \mid u_i^l \cdot \omega = \omega\}$ has nonempty interior.
- (3) If there exists $l \in \mathbb{N}$ such that for any $(i, j) \in \Omega_A$ we have $B_{i,j} = l \cdot A_{i,j}$, then for any $\omega \in X_{q_i}$ we have that $u_i^l \cdot \omega = \omega$, so that these sets have nonempty interior.

Remark 7.14. Notice that if $\omega = s_{i_1, i_2, n_1} s_{i_2, i_3, n_2} \cdots s_{i_k, i_{k+1}, n_k} \cdots$ of X_A is fixed under the action of $u_{i_1}^l$ ($l \in \mathbb{Z}$), then so are all the elements $\omega' = s_{i_1, i_2, m_1} s_{i_2, i_3, m_2} \cdots s_{i_k, i_{k+1}, m_k} \cdots$ for any choice of elements $1 \leq m_j \leq A_{i_j, i_{j+1}}$ ($j \geq 1$). Thus, the set $\{\omega \in X_A \mid u_i^l \cdot \omega = \omega\}$ is not a singleton in general. Also notice that, by Proposition 7.15, if the action of u_i^l for all $1 \leq i \leq N, l \in \mathbb{Z}$ is topologically free, then the matrix B satisfies Condition (E).

Now, we will look at topological freeness for this particular kind of actions.

Proposition 7.15. *Given $t = u_i^l$ for $1 \leq i \leq N$ and $l \in \mathbb{Z}$, and $\omega = s_{i_1, i_2, n_1} s_{i_2, i_3, n_2} \cdots s_{i_k, i_{k+1}, n_k} \cdots$ a fixed element by t . Then, the following are equivalent:*

- (1) $\text{Int}(\{\delta \in X_A \mid t \cdot \delta = \delta\}) = \emptyset$.
- (2) For every $n \geq 1$ there exist $m \geq n$ and j_{m+1} with:
- (a) $(i_m, j_{m+1}) \in \Omega_A$.
 - (b) $K_{m+1} = K_m \cdot \frac{B_{i_m, j_{m+1}}}{A_{i_m, j_{m+1}}} \notin \mathbb{Z}$.

Proof. We need to characterize when, for any $n \geq 1$, W_n^ω is not contained in $\{\delta \in X_A \mid t \cdot \delta = \delta\}$.

Since ω is fixed by t , by Lemma 7.12 we have

$$K_j = l \cdot \prod_{t=1}^j \frac{B_{i_t, i_{t+1}}}{A_{i_t, i_{t+1}}} \in \mathbb{Z}$$

for all $j \geq 0$. Thus, in order to get the condition above, for any $n \in \mathbb{N}$ we need to have $m \geq n$ and an alternative path

$$\gamma = s_{i_1, i_2, n_1} \cdots s_{i_{m-1}, i_m, n_{m-1}} s_{i_m, j_{m+1}, p_m} \cdots$$

such that $K_{m+1} = K_m \cdot \frac{B_{i_m, j_{m+1}}}{A_{i_m, j_{m+1}}} \notin \mathbb{Z}$. Hence, the equivalence is clear. \square

A practical situation under which Proposition 7.15 works is the following

Corollary 7.16. *Given $t = u_i^l$ for $1 \leq i \leq N$ and $l \in \mathbb{Z}$, we have that $\text{Int}(\{\omega \in X_A \mid t \cdot \omega = \omega\}) = \emptyset$ (where ω denotes $s_{i_1, i_2, n_1} s_{i_2, i_3, n_2} \cdots s_{i_k, i_{k+1}, n_k} \cdots$) if:*

- (1) $B_{i_j, i_{j+1}} \neq 0$ for all j .
- (2) For every $n, r \geq 1$ there exist a sequence $j_{n+1}, j_{n+2}, \dots, j_{n+r}$ with:
- (a) $(j_{n+t}, j_{n+t+1}) \in \Omega_A$ for all t .
 - (b) $\lim_{r \rightarrow \infty} \prod_{t=1}^r \left(\frac{B_{j_{n+t}, j_{n+t+1}}}{A_{j_{n+t}, j_{n+t+1}}} \right) = 0$.

Proof. Notice that, given n , we have that K_n is a fixed integer. So, by taking a sufficiently large sequence we can guarantee that $\frac{1}{K_n} > \prod_{t=1}^r \frac{B_{j_{n+t}, j_{n+t+1}}}{A_{j_{n+t}, j_{n+t+1}}}$, so that $K_{n+r} \notin \mathbb{Z}$. \square

An example of application of Corollary 7.16 is the following

Example 7.17. Suppose that B satisfies Condition (E) and that for every $1 \leq i \leq N$ we have that $|B_{i,i}| < |A_{i,i}|$. Then Corollary 7.16 applies for every $t = u_i^l$ for $1 \leq i \leq N$ and $l \in \mathbb{Z}$, because we can fix $j_{n+t} = i_n$ for all t , and then

$$\lim_{r \rightarrow \infty} \prod_{t=1}^r \left(\frac{B_{j_{n+t}, j_{n+t+1}}}{A_{j_{n+t}, j_{n+t+1}}} \right) = \lim_{r \rightarrow \infty} \left(\frac{B_{i_n, i_n}}{A_{i_n, i_n}} \right)^r = 0.$$

So, the action of α restricted to elements of the form $t = u_i^l$ for $1 \leq i \leq N$ and $l \in \mathbb{Z}$ is topologically free. In particular, under the additional hypothesis of B satisfying Condition (E), Corollary 7.16 applies for Katsura's conditions (2.4(2)) [15, Proposition 2.10].

Lemma 7.18. *Given an element $\omega = s_{i_1, i_2, n_1} s_{i_2, i_3, n_2} \cdots s_{i_k, i_{k+1}, n_k} \cdots$ of X_A , the following are equivalent:*

- (1) ω is fixed under the action of $s_I u_{t(I)}^l$ ($l \in \mathbb{Z}$ and I multiindex).
- (2) We have that:
 - (i) s_I is a cycle of the form $s_{i_1, i_2, n_1} s_{i_2, i_3, n_2} \cdots s_{i_k, i_{k+1}, n_k}$ for some $k \in \mathbb{N}$.
 - (ii) There exists a sequence $(s_I^{(j)})_{j \geq 1}$ of cycles with $s_I^{(0)} = s_I$, and a sequence $(T_j)_{j \geq 0}$ of integers with $T_0 = l$, such that $u_{t(I)}^{T_j} s_I^{(j)} = s_I^{(j+1)} u_{t(I)}^{T_{j+1}}$ for all $j \geq 1$, and $\omega = s_I s_I^{(1)} \cdots s_I^{(k)} \cdots$.

Proof. Sufficiency is clear. Let us proof necessity. Since $\omega = s_I u_{t(I)}^l \cdot \omega$, we have that $\omega = s_I \cdot \omega(1)$, whence (i) holds. Now,

$$s_I \cdot \omega(1) = \omega = s_I u_{t(I)}^l \cdot \omega = s_I u_{t(I)}^l s_I \cdot \omega(1),$$

so that $\omega(1) = u_{t(I)}^l s_I \cdot \omega(1) = s_I^{(1)} u_{t(I)}^{T_1} \omega(1) = s_I^{(1)} \cdot \omega(2)$. Recurrence on this argument proves that (ii) holds, so we are done. \square

Remark 7.19. Notice that, as it occurs in Lemma 7.7, the element ω fixed by $t = s_I u_{t(I)}^l$ is uniquely determined by t , and thus $\{\delta \in X_A \mid t \cdot \delta = \delta\} = \{\omega\}$.

Proposition 7.20. *Given $t = s_I u_i^l$ for $I \in E_A^*$, $1 \leq i \leq N$ and $l \in \mathbb{Z}$, and $\omega = s_{i_1, i_2, n_1} s_{i_2, i_3, n_2} \cdots s_{i_k, i_{k+1}, n_k} \cdots$ a fixed element by t . Then, the following are equivalent:*

- (1) $\text{Int}(\{\delta \in X_A \mid t \cdot \delta = \delta\}) = \emptyset$.
- (2) The graph E_A satisfies Condition (L).

Proof. Essentially, the proof is analog to that of Proposition 7.8.

(1) \Rightarrow (2) Suppose that E_A do not satisfy Condition (L). Then, there exists a cycle without exits s_I . In particular, for $i = s(I)$ and any $l \in \mathbb{Z}$ we have that $u_i^l s_I = s_I u_i^{T_l}$ for some $T_l \in \mathbb{Z}$. Hence, the element $\omega = s_I s_I \cdots$ is fixed by $t = s_I u_i^l$, and moreover for any $n \in \mathbb{N}$ we have that $W_n^\omega = \{\omega\}$. Thus, (1) fails.

(2) \Rightarrow (1) Suppose that $\omega = s_I s_I^{(1)} \cdots s_I^{(k)} \cdots$ is fixed by $t = s_I u_i^l$ for some $l \in \mathbb{Z}$. If E_A satisfies Condition (L) and $s_I = s_{i_1, i_2, n_1} s_{i_2, i_3, n_2} \cdots s_{i_k, i_{k+1}, n_k}$, there exists $1 \leq m \leq k$ and $j \in \Omega(i_m) \setminus \{i_{m+1}\}$. Hence, given any $n \in \mathbb{N}$ there is r large enough such that $r \cdot |I| > n$, and

the cycle $s_I^{(k+1)}$ has an exit $s_{i_m,j,p}$ with $1 \leq p \leq A_{i_m,j}$. For any infinite path $\gamma \in X_{p_j}$ we have that

$$\tau := s_I s_I^{(1)} \cdots s_I^{(r)} s_{i_1,i_2,n_1} s_{i_2,i_3,n_2} \cdots s_{i_{m-1},i_m,n_{m-1}} s_{i_m,j,p} \gamma \in W_n^\omega,$$

and by Remark 7.19 τ is not fixed by t . Thus (1) holds. \square

Remark 7.21. For the case of elements $s = s_I u_{r(I)}^t s_J^*$, if $\omega \in X_A$ and $s \cdot \omega = \omega$, then it is clear that $\omega = s_J \cdot \eta$ for some $\eta \in X_A$, and also that $s_J^* s_I \neq 0$, whence either $s_I = s_J s_K$ or $s_J = s_I s_K$ for a suitable multiindex K . In the first case, $s \cdot \omega = \omega$ if and only if $\eta = s_K u_{r(I)}^t \cdot \eta$. In the second case, $s \cdot \omega = \omega$ if and only if $\eta = s_{\hat{K}} u_{r(\hat{K})}^l \cdot \eta$, where $s_{\hat{K}} u_{r(\hat{K})}^l = u_{r(I)}^{-t} s_K$. So, in any case, the problem reduces to applying Lemma 7.18.

Hence, we conclude that

Theorem 7.22. *Let α be the action of $\mathcal{S}^{A,B}$ on X_A , and let $\mathcal{G}_{\Lambda_{A,B}}$ the associated groupoid. The following are equivalent:*

- (1) (a) *The graph E_A satisfies Condition (L).*
 (b) *The matrix B satisfies Condition (E).*
 (c) *For any fixed point $\omega = s_{i_1,i_2,n_1} s_{i_2,i_3,n_2} \cdots s_{i_k,i_{k+1},n_k} \cdots$ and every $n \geq 1$ there exist $m \geq n$ and j_{m+1} with:*
 - (i) $(i_m, j_{m+1}) \in \Omega_A$.
 - (ii) $K_{m+1} = K_m \cdot \frac{B_{i_m,j_{m+1}}}{A_{i_m,j_{m+1}}} \notin \mathbb{Z}$.
- (2) *The groupoid $\mathcal{G}_{\Lambda_{A,B}}$ is essentially principal.*

Proof. By Proposition 7.8, Proposition 7.15, Proposition 7.20 and Remark 7.21, (1) is equivalent to the action of $\mathcal{S}^{A,B}$ being topologically free. By Remark 7.6 and Theorem 7.5, this is equivalent to the groupoid $\mathcal{G}_{\Lambda_{A,B}}$ being essentially principal, as desired. \square

And as a practical consequence:

Proposition 7.23. *Let α be the action of $\mathcal{S}^{A,B}$ on X_A , and let $\mathcal{G}_{\Lambda_{A,B}}$ the associated groupoid. If*

- (1) *The graph E_A satisfies Condition (L).*
- (2) *The matrix B satisfies Condition (E).*
- (3) *For any fixed point $\omega = s_{i_1,i_2,n_1} s_{i_2,i_3,n_2} \cdots s_{i_k,i_{k+1},n_k} \cdots$ and for every $n, r \geq 1$ there exist a sequence $j_{n+1}, j_{n+2}, \dots, j_{n+r}$ with:*
 - (a) $(j_t, j_{t+1}) \in \Omega_A$ for all t .
 - (b) $\lim_{r \rightarrow \infty} \prod_{t=1}^r \left(\frac{B_{j_{n+t}, j_{n+t+1}}}{A_{j_{n+t}, j_{n+t+1}}} \right) = 0$.

then the groupoid $\mathcal{G}_{\Lambda_{A,B}}$ is essentially principal.

8. SIMPLICITY OF $\mathcal{O}_{A,B}$

In this section we use the results in the previous sections to characterize the simplicity of $\mathcal{O}_{A,B}$. Notice that, by Theorem 4.9, it is enough to show simplicity for $C_r^*(\mathcal{G}_{\Lambda_{A,B}})$. The central result is

Theorem 8.1. *Consider the initial matrices A, B . If the matrix B satisfies Condition (E), then the following are equivalent:*

- (1) (a) *The matrix A is irreducible.*
 (b) *The graph E_A satisfies Condition (L).*
 (c) *For any fixed point $\omega = s_{i_1, i_2, n_1} s_{i_2, i_3, n_2} \cdots s_{i_k, i_{k+1}, n_k} \cdots$ and every $n \geq 1$ there exist $m \geq n$ and j_{m+1} with:*
 - (i) $(i_m, j_{m+1}) \in \Omega_A$.
 - (ii) $K_{m+1} = K_m \cdot \frac{B_{i_m, j_{m+1}}}{A_{i_m, j_{m+1}}} \notin \mathbb{Z}$.
- (2) $\mathcal{O}_{A, B}$ *is simple.*

Proof. By Remark 3.16 and Corollary 5.16 the groupoid $\mathcal{G}_{\Lambda_{A, B}}$ is Hausdorff, étale with second countable unit space. By Corollary 6.8 and Theorem 7.22, conditions in point (1) are equivalent to $\mathcal{G}_{\Lambda_{A, B}}$ being a minimal essentially principal groupoid. These three facts jointly are equivalent to $C_r^*(\mathcal{G}_{\Lambda_{A, B}})$ being simple [4, Theorem 5.1]. Thus, Theorem 4.9 give us the desired result. \square

And as a direct consequence we have:

Proposition 8.2. *Consider the initial matrices A, B . If*

- (1) *The matrix A is irreducible.*
- (2) *The graph E_A satisfies Condition (L).*
- (3) *The matrix B satisfies Condition (E).*
- (4) *For any fixed point $\omega = s_{i_1, i_2, n_1} s_{i_2, i_3, n_2} \cdots s_{i_k, i_{k+1}, n_k} \cdots$ and for every $n, r \geq 1$ there exist a sequence $j_{n+1}, j_{n+2}, \dots, j_{n+r}$ with:*
 - (a) $(j_t, j_{t+1}) \in \Omega_A$ *for all* t .
 - (b) $\lim_{r \rightarrow \infty} \prod_{t=1}^r \left(\frac{B_{j_{n+t}, j_{n+t+1}}}{A_{j_{n+t}, j_{n+t+1}}} \right) = 0$.

then $\mathcal{O}_{A, B}$ is simple.

Proof. Replace Theorem 7.22 by Proposition 7.23 in the proof of Theorem 8.1. \square

As a corollary we have

Corollary 8.3. *Consider the initial matrices A, B . If they satisfy (2.4(2)) and B satisfies Condition (E), then $\mathcal{O}_{A, B}$ is simple.*

Remark 8.4. Notice that we need an extra property –Condition (E)– to obtain a characterization result for simplicity of $\mathcal{O}_{A, B}$, so that our results do not give a complete characterization of simplicity for $\mathcal{O}_{A, B}$. Nevertheless, our strategy will allow us to describe simplicity of $\mathcal{O}_{A, B}$ for a broad collection of algebras, including the ones given by Katsura. Moreover, the results are obtained in a more natural way, by linking this property to dynamical properties of X_A .

9. PURE INFINITENESS OF $\mathcal{O}_{A, B}$

In this section we use the results in the previous sections to give sufficient conditions for the algebra $\mathcal{O}_{A, B}$ being purely infinite (simple). As in Section 8 notice that, by Theorem 4.9, it is enough to show simplicity for $C_r^*(\mathcal{G}_{\Lambda_{A, B}})$. For this we need to recall a slight reformulation of a definition from [1].

Definition 9.1 ([1, Definition 2.1]). We say that a second countable étale groupoid \mathcal{G} is locally contracting in for every nonempty open subset U of $\mathcal{G}^{(0)}$ there exists an open subset V in U and an slice S such that $\overline{V} \subset S^{-1}S$ and $S\overline{V}S^{-1}$ is properly contained in V .

In our case, using [9, Proposition 4.18] and (5.15) we have that, for any $s \in \mathcal{S}^{A,B}$ and for any $U \subseteq X_{s^*s}$, the sets $\Omega(s, U) = \{[s, x] \in \mathcal{G}_{\Lambda_{A,B}} \mid x \in U\}$ are slices (in fact they form a basis for the topology of $\mathcal{G}_{\Lambda_{A,B}}$). In view of that, we can restrict our attention to this kind of slices. Given any such slice $T := \Omega(s, U)$, it is easy to see that for any $x \in \overline{V} \subset T^{-1}T$ the unique element of T which can act by conjugation on x is $[s, x]$, and so $[s, x] \cdot x \cdot [s, x]^{-1} = s \cdot x$. Under this reduction, we can understand $T\overline{V}T^{-1}$ as the set $\alpha_s(\overline{V})$ of images of \overline{V} by the action α_s performed by the element $s \in \mathcal{S}^{A,B}$ defining the slice. Also, recall that $\{W_n^\gamma \mid n \in \mathbb{N}, \gamma \in X_A\}$ is a basis of clopen sets of X_A .

9.2. Thus, in order to check local contractiveness of $\mathcal{G}_{\Lambda_{A,B}}$ it is enough to show that for any $n \in \mathbb{N}$ and any $\gamma \in X_A$ there exist $n < m \in \mathbb{N}$, $\delta \in X_A$ and $s \in \mathcal{S}^{A,B}$ such that:

- (1) $W_m^\delta \subseteq W_n^\gamma$ (equivalently, $\delta|_n = \gamma|_n$).
- (2) $W_m^\delta \subseteq X_{s^*s}$.
- (3) For any $\omega \in W_m^\delta$ we have $\alpha_s(\omega) = s \cdot \omega \in W_m^\delta$.
- (4) There exists $\hat{\omega} \in W_m^\delta$ such that $\hat{\omega} \neq \alpha_s(\omega) = s \cdot \omega$ for every $\omega \in W_m^\delta$.

A subsequent reduction can be done on the choice of $s \in \mathcal{S}^{A,B}$. To be concrete, we only need to analyze the following cases:

- (1) $s = u_i$, for any $1 \leq i \leq N$.
- (2) $s = u_i^*$, for any $1 \leq i \leq N$.
- (3) $s = s_I$ for any multiindex I .
- (4) $s = s_I^*$ for any multiindex I .

because any $s \in \mathcal{S}^{A,B}$ can be written using the above elements. Let us study case by case.

Lemma 9.3. *Let $n \in \mathbb{N}$, $\gamma \in X_A$. Then, $\alpha_{u_i}(W_n^\gamma) \subseteq W_n^\gamma$ implies that $s(\gamma|_n) = i, r(\gamma|_n) = j$ and there exists $l \in \mathbb{Z}$ such that $u_i(\gamma|_n) = (\gamma|_n)u_j^l$. In particular, the above inclusion is always an equality.*

Proof. Since $u_i = u_i p_i$, it is clear that $s(\gamma|_n) = i$. Now, set $r(\gamma|_n) = j$. By definition, for any $\omega \in W_n^\gamma$ we have $\omega|_n = \gamma|_n$. Hence, $u_i \cdot \omega \in W_n^\gamma$ means that for any $\delta \in X_{p_j}$ there exists $l \in \mathbb{Z}$ such that $u_i(\gamma|_n) \cdot \delta = ((\gamma|_n)u_j^l) \cdot \delta$.

In order to see that the above inequality is an equality it is enough to observe that, for any $\delta \in X_{p_j}$, the element $\eta := (u_i^*)^l \cdot \delta \in X_{p_j}$ and $u_i(\gamma|_n) \cdot \eta = ((\gamma|_n)u_j^l) \cdot \delta$. \square

It is clear that the same argument applies to $s = u_i^*$. So, in order to check local contractiveness of $\mathcal{G}_{\Lambda_{A,B}}$ we only need to pay attention to the remaining cases.

Lemma 9.4. *Let $n \in \mathbb{N}$, $\gamma \in X_A$. Then, for any $n = k \cdot |I|$, $\alpha_{s_I}(W_n^\gamma) \subseteq W_n^\gamma$ implies that $s(\gamma|_n) = r(I)$, s_I is a cycle and $\gamma|_n = s_I s_I \cdots s_I$ (k times). Moreover, the above inclusion is strict if and only if the cycle s_I has an exit.*

Proof. It is obvious that $s(\gamma|_n) = r(I)$. On the other side, if for any $\omega \in W_n^\gamma$ we have $s_I \cdot \omega \in W_n^\gamma$, then $r(I) = s(s_I(\gamma|_n)) = s(I)$, whence s_I is a cycle. Also, $(s_I \gamma|_n)|_n = \gamma|_n$, so that $\gamma|_n = s_I \tau$. Recurrence on this argument shows that $\gamma|_n = s_I s_I \cdots s_I$.

Now, if s_I has no exits, then any $\omega \in X_{p_r(I)}$ must be $s_I s_I s_I \cdots$, and so we have equality. Conversely, if s_I have an exit, then there exists a multiindex $K \neq I$ such that $s_I s_K \neq 0$. Then, if $r(K) = j$, for any $\delta \in X_{p_r(K)}$ we have that $(\gamma|_n) s_K \cdot \delta \in W_n^\gamma$ but not in $\alpha_{s_I}(W_n^\gamma)$, so we are done. \square

Since $\alpha_{s_I^*}$ is a partial inverse of α_{s_I} , we conclude the following

Lemma 9.5. *Let $n \in \mathbb{N}$, $\gamma \in X_A$. Then, $\alpha_{s_I^*}(W_n^\gamma) \subseteq W_n^\gamma$ implies that s_I is a cycle with no exits, and so the above inclusion is always an equality.*

Proposition 9.6. *If every finite path in the graph E_A can be enlarged to a cycle and E_A satisfies Condition (L), then $\mathcal{G}_{\Lambda_{A,B}}$ is locally contracting.*

Proof. Let $n \in \mathbb{N}$ and $\gamma \in X_A$. By hypothesis there exists a finite path $\delta \in E_A$ with $s(\delta) = r(\gamma|_n)$ and $r(\delta) = s(\gamma|_n)$. If the length of δ is m , we have $s = \omega = (\gamma|_n)\delta$ is a cycle and $W_{n+m}^\omega \subset W_n^\gamma$. Since E_A satisfies Condition (L), s has an exit, so that by Lemma 9.4 $\alpha_s(W_{n+m}^\omega) \subsetneq W_{n+m}^\omega$. Thus, the results holds by (9.2). \square

So, we can prove the main result of this section

Theorem 9.7. *Consider the initial matrices A, B . If*

- (1) *The matrix A is irreducible.*
- (2) *The graph E_A satisfies Condition (L).*
- (3) *The matrix B satisfies Condition (E).*
- (4) *For any fixed point $\omega = s_{i_1, i_2, n_1} s_{i_2, i_3, n_2} \cdots s_{i_k, i_{k+1}, n_k} \cdots$ and every $n \geq 1$ there exist $m \geq n$ and j_{m+1} with:*
 - (a) $(i_m, j_{m+1}) \in \Omega_A$.
 - (b) $K_{m+1} = K_m \cdot \frac{B_{i_m, j_{m+1}}}{A_{i_m, j_{m+1}}} \notin \mathbb{Z}$.

then $\mathcal{O}_{A,B}$ is purely infinite simple.

Proof. By Theorem 8.1 the algebra $\mathcal{O}_{A,B}$ is simple. Since the matrix A is irreducible, the graph E_A is transitive (i.e. there exists a path between any two vertices of E_A). Thus, $\mathcal{G}_{\Lambda_{A,B}}$ is locally contracting by Proposition 9.6. By [1, Proposition 2.4] every nonzero hereditary sub- C^* -algebra of $C_r^*(\mathcal{G}_{\Lambda_{A,B}})$ contain an infinite projection, and then so does $\mathcal{O}_{A,B}$ by Theorem 4.9. Hence, $\mathcal{O}_{A,B}$ is purely infinite simple, as desired. \square

Hence, under the hypotheses of either Theorem 8.1, Proposition 8.2 or Corollary 8.3, we conclude that $\mathcal{O}_{A,B}$ is purely infinite simple. This includes the case considered by Katsura, when Condition (E) is satisfied. Also, since A irreducible plus Condition (L) implies Condition (K), Theorem 9.7 becomes an extension of [12, Theorem 16.2] to the case of B being a nonzero matrix.

Finally, we will show that, under Condition (E), it is possible to show a partial version of [16, Proposition 4.5].

Lemma 9.8. *Let G_0, G_1 be finitely generated abelian groups. Then, there exist $N \in \mathbb{N}$, $A \in M_N(\mathbb{Z}^+)$, $B \in M_N(\mathbb{Z})$ such that:*

- (1) *The matrix B satisfies Condition (E).*
- (2) *The matrix A is irreducible.*

- (3) $A_{i,i} \geq 2$ and $B_{i,i} = 1$ for every $1 \leq i \leq N$.
 (4) $G_0 \cong \text{coker}(I - A) \oplus \ker(I - B)$ and $G_1 \cong \text{coker}(I - B) \oplus \ker(I - A)$.

Proof. By using the results in [15, Section 3], we can guarantee that there exists $N' \in \mathbb{N}$, $A' \in M_{N'}(\mathbb{Z}^+)$, $B' \in M_{N'}(\mathbb{Z})$ satisfying (2-4). We can choose A', B' such that the Smith Normal Form of both $I' - A'$ and $I' - B'$ are nonzero. Hence, by performing elementary row and column operations, we can replace these matrices by $A'' \in M_{N'}(\mathbb{Z}^+)$, $B'' \in M_{N'}(\mathbb{Z})$ such that $0 \neq A''_{i,j}, I' - B''_{i,j}$ for every $1 \leq i, j \leq N'$. Thus, $A'', C := I' - B''$ satisfies (1) and (2), while:

- (1) $\text{coker}(I - A) \cong \text{coker}(I' - A'')$ and $\ker(I - A) \cong \ker(I' - A'')$
 (2) $\text{coker}(C) \cong \text{coker}(I' - B'')$ and $\ker(C) \cong \ker(I' - B'')$

Now define $N = 2N'$, and define matrices $A \in M_N(\mathbb{Z}^+)$, $B \in M_N(\mathbb{Z})$ as follows:

$$A = \begin{pmatrix} 2I' & A'' \\ I' & 2I' \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} I' & C \\ I' & I' \end{pmatrix}.$$

Since A'' is irreducible, the graph $E_{A''}$ is transitive. It is easy to see that then so is E_A , whence A is irreducible. Thus, A and B satisfy (1-3). On one side

$$I - A = \begin{pmatrix} I' & I' \\ 0 & I' \end{pmatrix} \cdot \begin{pmatrix} A'' - I' & 0 \\ 0 & -I' \end{pmatrix} \cdot \begin{pmatrix} 0 & -I' \\ I' & I' \end{pmatrix}.$$

Hence, $\text{coker}(I - A) \cong \text{coker}(I' - A')$ and $\ker(I - A) \cong \ker(I' - A')$. On the other side

$$I - B = \begin{pmatrix} C & 0 \\ 0 & I' \end{pmatrix} \cdot \begin{pmatrix} 0 & -I' \\ -I' & 0 \end{pmatrix}.$$

Hence, $\text{coker}(I - B) \cong \text{coker}(I' - B')$ and $\ker(I - B) \cong \ker(I' - B')$. Thus, (4) is fulfilled, so we are done. \square

As a consequence of Theorem 9.7 and Lemma 9.8, we conclude the following restricted version of [16, Proposition 4.5].

Theorem 9.9 (c.f. [16, Proposition 4.5]). *Let G_0, G_1 be finitely generated abelian groups. Then, there exist $N \in \mathbb{N}$, $A \in M_N(\mathbb{Z}^+)$, $B \in M_N(\mathbb{Z})$ satisfying Condition (E), such that:*

- (1) $\mathcal{O}_{A,B}$ is unital Kirchberg algebra.
 (2) $K_i(\mathcal{O}_{A,B}) \cong G_i$ for $i = 0, 1$.

Notice that Theorem 9.9 means that we can represent any unital Kirchberg algebra (up to isomorphism) as a Katsura algebra $\mathcal{O}_{A,B}$ such that the matrix B satisfies Condition (E), and thus as the groupoid C^* -algebra of a minimal essentially principal locally contracting groupoid $\mathcal{G}_{\Lambda_{A,B}}$.

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